Statistical Energy Analysis

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Contents:

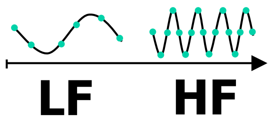
Reference: Foundation of statistical energy analysis in vibroacoustics, A. Le Bot, Oxford University Press, may 2015.

Up2HF, summer school, Celya, 1-3 july 2015

Introduction

When the frequency increases,

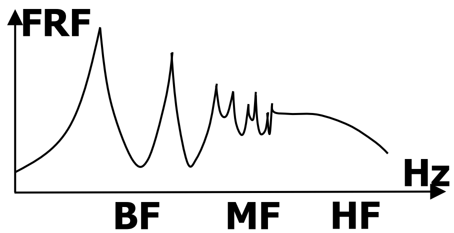
- Huge number of dof



Frequency limitation of FEM

Frequency limits by FEM

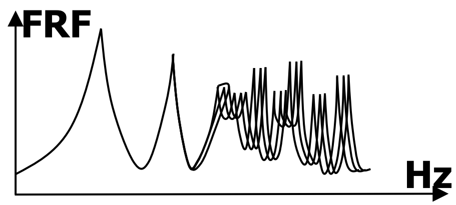
automobile	**< 1000 Hz**
aircraft	**< 100 Hz**
ship	**< 10 Hz**
Launcher, building	**< 1 Hz**



- The modal density increases (!)

$$n=\frac{\Delta N}{\Delta \omega}$$

Inefficiency of modal analysis

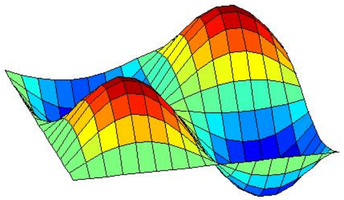


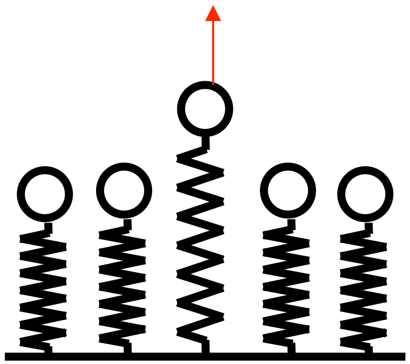
- High sensitivity to data

Poor predictibility

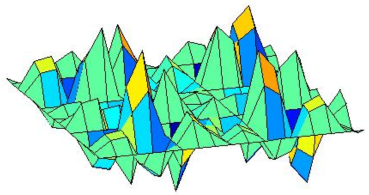
Statistical energy analysis is a statistical theory of sound and vibration when the number of modes is large and vibration is sufficiently disorganized.

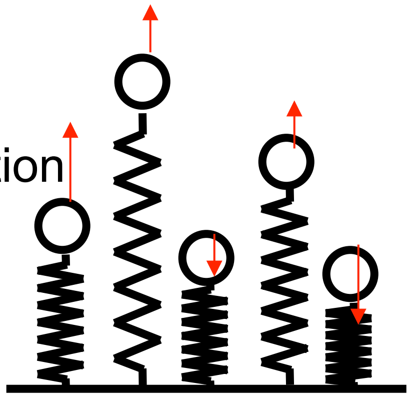
coherent vibration



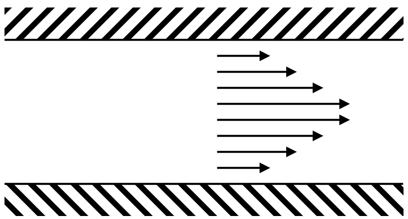


uncoherent vibration

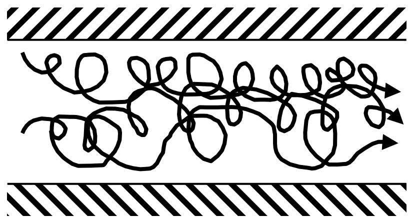




laminar flow



turbulent flow



modes <-> molecules

diffuse field <-> thermal equilibrium

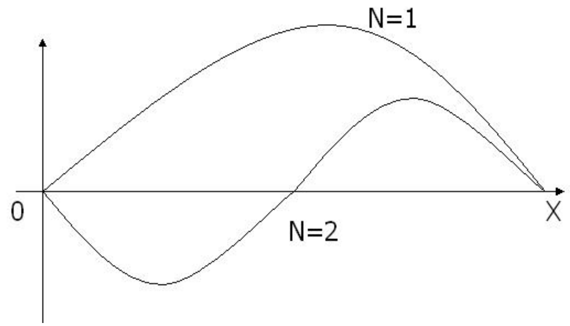
Modal density

Each finite structure (or bounded room) has a sequence of natural frequencies which tends to infinite.

Ex. string: $\omega_i = i\dfrac{\pi}{L}$ and $\psi_i(x) = \sin\left(i\pi\dfrac{x}{L}\right)$ for $i = 1, 2, ...$

Dimension 1

Resonance = integer number of half-wavelengths



wavenumber **length** **#modes**

wavenumber $\dfrac{\kappa}{\pi}L = N$

dispersion relationship $\kappa(\omega)$

modal density $n(\omega) = \dfrac{dN}{d\omega} = \dfrac{L}{\pi}\dfrac{d\kappa}{d\omega}$

group speed $c_g = \dfrac{d\omega}{d\kappa}$

1D modal density **length**

$$\boxed{n(\omega) = \frac{L}{\pi c_g}}$$

group speed

Dimension 2

Resonance = integer number of half-wavelengths in each direction

wavenumber vector

$$\kappa = (\kappa_x, \kappa_y)$$

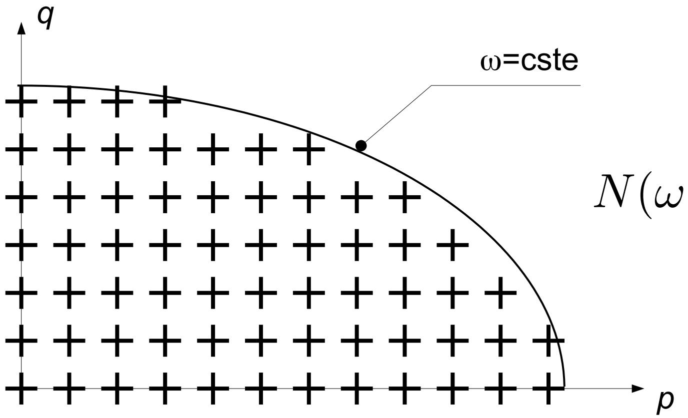
length integer width integer

$$\frac{\kappa_x}{\pi} a = p \qquad \frac{\kappa_y}{\pi} b = q \qquad (p,q) \in \mathbb{N}^2$$

number of modes below ω:

$$N(\omega) = \mathrm{Card}\left\{ (p,q) \in \mathbb{N}^2, \left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2 \leq \kappa^2(\omega) \right\}$$

$N(\omega)$ is approximated by the area under the ellipse $1 = \left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2$



$$N(\omega) = \frac{\pi}{4} \frac{a\kappa}{\pi} \frac{b\kappa}{\pi} = \frac{ab\kappa^2}{4\pi}$$

modal density $n(\omega) = \frac{dN}{d\omega} = \frac{ab}{2\pi} \kappa \frac{d\kappa}{d\omega}$

group speed $c_g = \frac{d\omega}{d\kappa}$ phase speed $c_p = \frac{\omega}{\kappa}$

2D modal density surface

$$\boxed{n(\omega) = \frac{S\omega}{2\pi c_p c_g}}$$

phase speed group speed

Dimension 3

Resonance = integer number of half-wavelengths in <u>each</u> direction

wavenumber vector

$$\kappa = (\kappa_x, \kappa_y, \kappa_z)$$

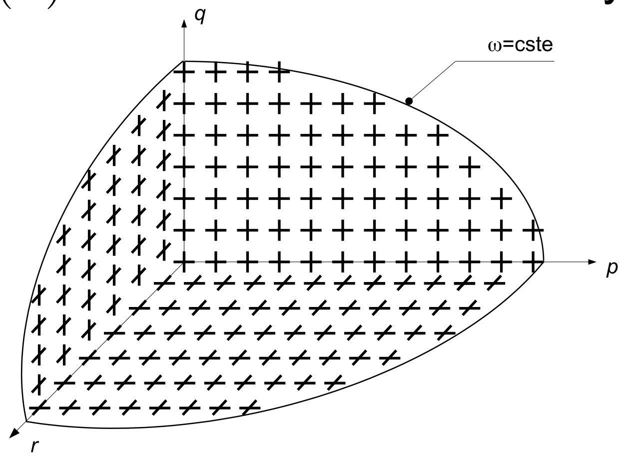
length integer width integer height integer

$$\frac{\kappa_x}{\pi} a = p \qquad \frac{\kappa_y}{\pi} b = q \qquad \frac{\kappa_z}{\pi} c = r \qquad (p, q, r) \in \mathbb{N}^3$$

number of modes below ω:

$$N(\omega) = \mathrm{Card} \left\{ (p, q, r) \in \mathbb{N}^3, \left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2 + \left(\frac{r\pi}{c}\right)^2 \leq \kappa^2(\omega) \right\}$$

$N(\omega)$ is the volume enclosed by the ellipsoid surface $1 = \left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2 + \left(\frac{r\pi}{c}\right)^2$



$$N(\omega) = \frac{1}{8} \times \frac{4}{3} \pi \frac{a\kappa}{\pi} \frac{b\kappa}{\pi} \frac{c\kappa}{\pi} = \frac{abc\kappa^3}{6\pi^2}$$

modal density $n(\omega) = \dfrac{dN}{d\omega} = \dfrac{abc}{2\pi^2} \kappa^2 \dfrac{d\kappa}{d\omega}$

group speed $c_g = \dfrac{d\omega}{d\kappa}$ phase speed $c_p = \dfrac{\omega}{\kappa}$

3D modal density surface

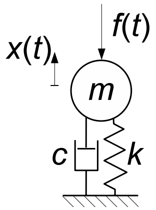
$$\boxed{n(\omega) = \frac{V\omega^2}{2\pi^2 c_p^2 c_g}}$$

phase speed group speed

Damping

At high frequencies, the vibrational level is controlled by damping

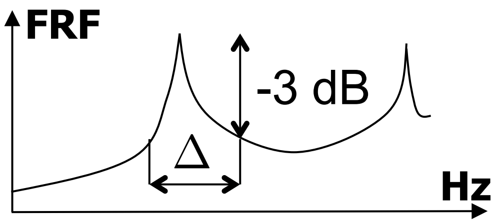
- viscous damping coefficient c
- modal damping ratio ζ
- damping loss factor η
- half-power bandwidth Δ



$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{2m\omega_0}$$

$$\eta = 2\zeta \qquad \Delta = \eta\omega_0$$

Half-power bandwidth



half-power bandwidth Δ

damping loss factor $\eta = \dfrac{\Delta}{\omega_0}$

This a low frequency method

Reverberation time

Tr=Time for a decrease of 60 dB of vibrational level (or SPL)

Time decrease of energy $E(t) = E_0 \exp\left(-\eta\omega t\right)$ $\quad 10\log_{10} E(T_r) - 10\log_{10} E_0 = -60$

In structures

$$\boxed{T_r = \frac{2.2 \times 2\pi}{\eta\omega}}$$

In rooms

$$\boxed{T_r = 0.16\frac{V}{\alpha S}}$$

(Sabine's formula)

Power balance

Steady state condition $\quad P_{inj} = \eta\omega E$

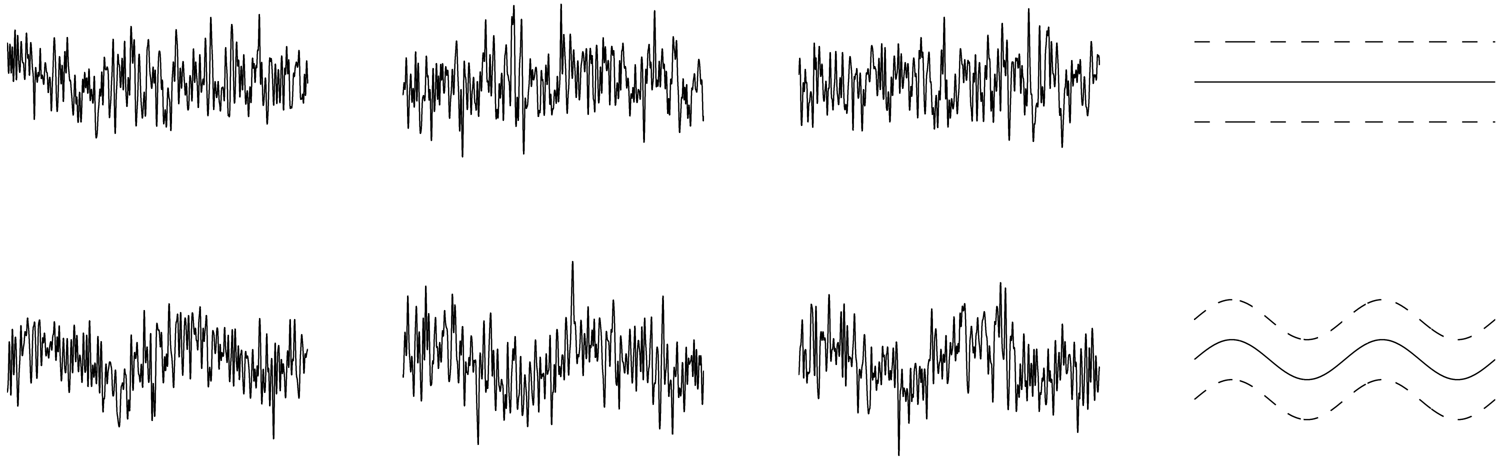
Measurement of P_{inj} with an impedance head

This is a high frequency method

Measurement of E with accelerometers or vibrometer

Random functions

A random function is a map $t \mapsto x(t)$ where $x(t)$ is a random variable



< . > probabilitic expectation

Auto-correlation – power spectral density

time time delay

$$\langle x(t)x(t+\tau)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\exp{(i\omega t)}d\omega$$

Auto-correlation **Power spectral density (PSD)**

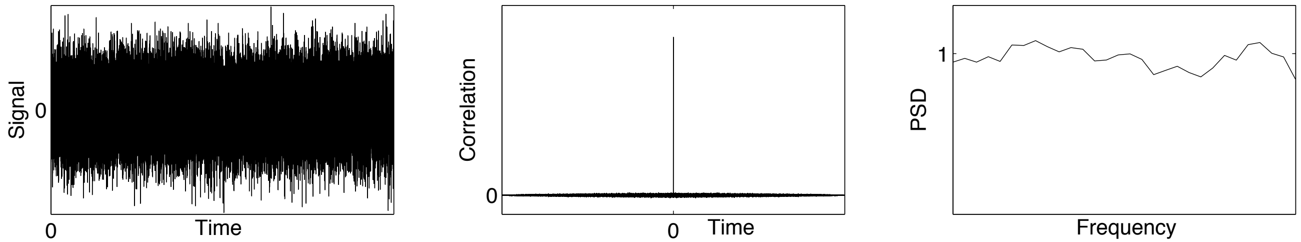
At $\tau = 0$

$$\boxed{\langle x^2\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)d\omega}$$

Stationarity

A random function is stationary if $\langle x(t) \rangle$ and $\langle x(t)x(t+\tau \rangle$ do not depend on *t.*

White noise

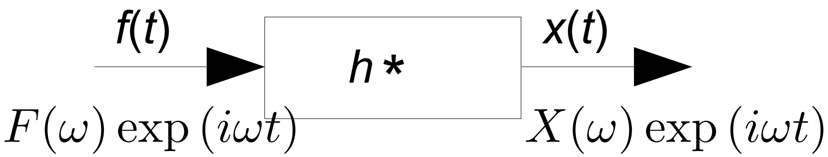


$$\begin{cases} S_{xx}(\omega) = S_0 \\ \langle x(t)x(t+\tau \rangle = \frac{S_0}{2\pi}\delta(t) \end{cases}$$

Uncorrelation

$x(t)$ and $y(t)$ are said uncorrelated if $\langle x(t)y(t+\tau) \rangle = 0$ or $S_{xy}(\omega) = 0$

Response to a linear system



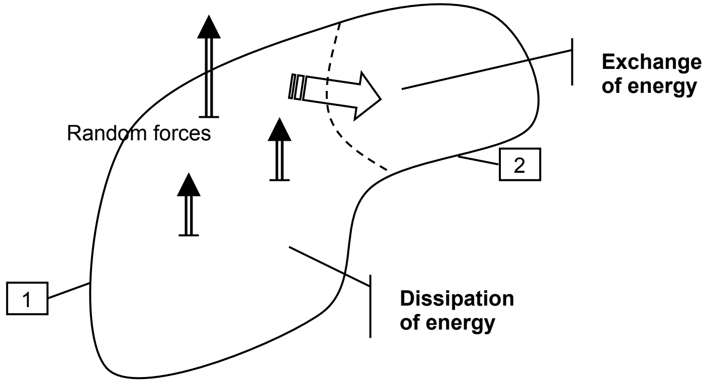
frequency response function $H(\omega) = \dfrac{X(\omega)}{F(\omega)}$

PSD of output

$$S_{xx}(\omega) = |H|^2(\omega)S_{ff}(\omega) \qquad S_{\dot{x}\dot{x}}(\omega) = \omega^2|H|^2(\omega)S_{ff}(\omega) \qquad S_{f\dot{x}}(\omega) = i\omega H(\omega)S_{ff}(\omega)$$

Modal approach of statistical energy analysis

The goal of statistical energy analysis is to provide an analysis of vibrating structures in terms of energy and power.



Flexible structure excited by random forces

E_i : vibrational energy in subsystem i.
K_i: kinetic energy in subsystem i.
V_i : elastic energy in subsystem i.

$$E_i = V_i + K_i$$

How is the vibrational energy shared between the elastic and kinetic forms?

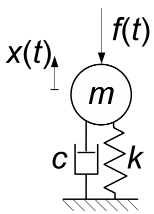
$P_{inj,i}$: power injected in subsystem i by external forces
$P_{diss,i}$: power dissipated in subsystem i.

How to compute the injected power from spectrum of forces?
Is the dissipated power related to vibrational energy?

P_{ij} : power exchanged between subsystem i and j.

Is there a relation between P_{ij} and E_i, E_j?

Single resonator



$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{c}{2m\omega_0}$$

Assumptions

- linear mechanical oscillator
- stationary random force f(t)
- white noise force of spectrum S_0

Governing equation

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Frequency response function

$$H(\omega) = \frac{1}{m\omega_0^2\left[1 + 2i\frac{\omega}{\omega_0} - \frac{\omega^2}{\omega_0^2}\right]}$$

Power balance

$$\underbrace{\frac{d}{dt}\left[\underbrace{\frac{1}{2}m\dot{x}^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}kx^2}_{\text{elastic energy}}\right]}_{\text{vibrational energy}} + \underbrace{c\dot{x}^2}_{\text{dissipated power}} = \underbrace{f\dot{x}}_{\text{injected power}}$$

i) Equality of kinetic and elastic energies

$$\langle V\rangle = \frac{1}{2}k\langle x^2\rangle = \frac{k}{4\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)d\omega = \frac{kS_0}{4\pi}\int_{-\infty}^{\infty}|H|^2(\omega)d\omega = \frac{kS_0}{8\zeta m^2\omega_0^3}$$

$$\boxed{\langle V\rangle = \langle K\rangle}$$

$$\langle K\rangle = \frac{1}{2}m\langle \dot{x}^2\rangle = \frac{m}{4\pi}\int_{-\infty}^{\infty} S_{\dot{x}\dot{x}}(\omega)d\omega = \frac{mS_0}{4\pi}\int_{-\infty}^{\infty}\omega^2|H|^2(\omega)d\omega = \frac{mS_0}{8\zeta m^2\omega_0}$$

ii) Injected power

$$\langle P_{inj}\rangle = \langle f\dot{x}\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{f\dot{x}}(\omega)d\omega = \frac{S_0}{2\pi}\int_{-\infty}^{\infty} i\omega H(\omega)d\omega = \frac{S_0}{2m}$$

$$\boxed{\langle P_{inj}\rangle = \frac{S_0}{2m}}$$

iii) Mean power balance

$$\langle\frac{dE}{dt}\rangle = \langle m\dot{x}\ddot{x}\rangle + k\langle x\dot{x}\rangle = \frac{m}{2\pi}\int_{-\infty}^{\infty} S_{\dot{x}\ddot{x}}(\omega)d\omega + \frac{k}{2\pi}\int_{-\infty}^{\infty} S_{x\dot{x}}(\omega)d\omega = \frac{mS_0}{2\pi}\int_{-\infty}^{\infty} i\omega^3|H|^2(\omega)d\omega + \frac{kS_0}{2\pi}\int_{-\infty}^{\infty} i\omega|H|^2(\omega)d\omega = 0$$

odd function

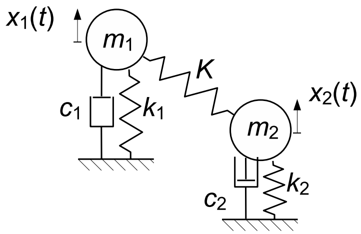
$$\langle P_{diss}\rangle = c\langle \dot{x}^2\rangle = \frac{2c}{m}\times\frac{1}{2}m\langle \dot{x}^2\rangle = \frac{2c}{m}\langle K\rangle = \frac{c}{m}\langle E\rangle$$

$$\boxed{\langle P_{diss}\rangle = \frac{c}{m}\langle E\rangle}$$

$$\langle\frac{dE}{dt}\rangle + \langle P_{diss}\rangle = \langle P_{inj}\rangle$$

$$\boxed{\langle P_{inj}\rangle = \frac{c}{m}\langle E\rangle}$$

Pair of resonators
Lyon and Sharton 1968



Assumptions

- linear mechanical oscillators
- stationary random forces
- white noises force of spectrum
- uncorrelated forces
- conservative coupling (inertial, gyroscopic and elastic)

Governing equation

$$\underbrace{\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}}_{\mathbf{C}} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} k_1 & -K \\ -K & k_2 \end{pmatrix}}_{\mathbf{K}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Frequency response matrix

$$\mathbf{H}(\omega) = \left[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right]^{-1}$$

Power balance

$$\frac{d}{dt}\left[\underbrace{\frac{1}{2}m_i\dot{x}_i^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}k_i x_i^2}_{\text{elastic energy}} - \frac{1}{2}K x_1 x_2\right] + \underbrace{c_i\dot{x}_i^2}_{\text{dissipated power}} + \underbrace{\frac{1}{2}K(x_i\dot{x}_j - \dot{x}_i x_j)}_{\text{exchanged power}} = \underbrace{f_i\dot{x}_i}_{\text{injected power}}$$

The elastic energy of the coupling has been shared between adjacent oscillators

i) Equality of kinetic and elastic energies $\boxed{\langle V_i\rangle = \langle K_i\rangle}$

It exists a unique way to share the coupling energy between oscillators to enforce the equality for each oscillator.

ii) Injected power $\boxed{\langle P_{inj,i}\rangle = \dfrac{S_i}{2m_i}}$

The injected power does not depend on the presence of a coupling

iii) Mean power balance
Since each product $\langle x_i^{(p)} x_j^{(q)}\rangle$ is a linear combination of the power spectrums S_i

$$\langle P_{ij}\rangle = \begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$$

and

$$\begin{pmatrix} \langle E_1\rangle \\ \langle E_2\rangle \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$$

$$\langle P_{ij}\rangle = \begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}^{-1} \begin{pmatrix} \langle E_1\rangle \\ \langle E_2\rangle \end{pmatrix}$$

So, by expanding $\langle P_{12}\rangle = B_1\langle E_{\rangle} - B_2\langle E_2\rangle$
After calculating all the integrals by the residue theorem

$$\boxed{\begin{aligned} &\langle P_{12}\rangle = B(\langle E_{\rangle} - \langle E_2\rangle) \\ &B = B_1 = B_2 = \frac{K^2(\Delta_1 + \Delta_2)}{m_1 m_2\left[(\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\Delta_1\omega_2^2 + \Delta_2\omega_1^2)\right]} \end{aligned}}$$

« blocked » natural frequency

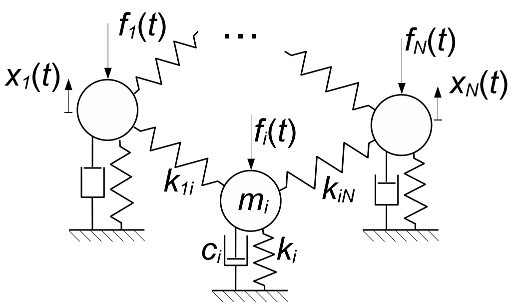
$$\omega_i = \sqrt{k_i/m_i}$$

half-power bandwidth

$$\Delta_i = c_i/m_i$$

Set of resonators

Newland 1966



Assumptions

- white noise forces of spectrum S_i for $i=1,\dots,n$
- uncorrelated forces
- conservative coupling
- light coupling

Governing equation

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F}(t)$$

Frequency response matrix

$$\mathbf{H}(\omega) = \left[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right]^{-1}$$

Perturbation technique $\epsilon \propto K$

$$\mathbf{K} = \begin{pmatrix} k_1 & \mathrm{O} \\ \mathrm{O} & k_n \end{pmatrix} + \epsilon \begin{pmatrix} 0 & \times \\ \times & 0 \end{pmatrix}$$

small parameter

$$x_i(t) = x_{i0}(t) + \epsilon x_{i1}(t) + \epsilon^2 x_{i2}(t) + o(\epsilon^2)$$

i) Equality of kinetic and elastic energies $\boxed{\langle V_i \rangle = \langle K_i \rangle}$

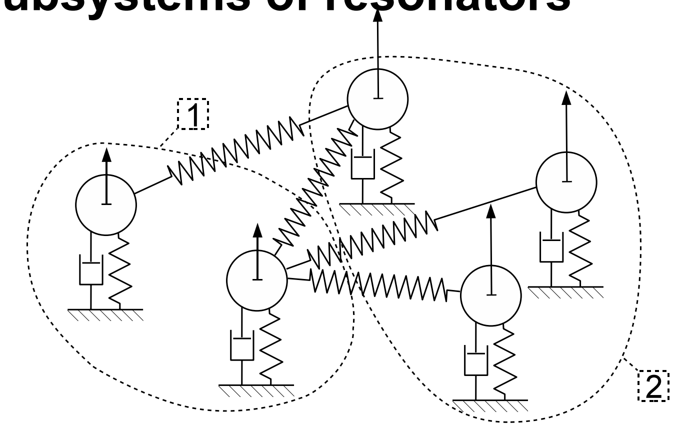
ii) Injected power $\boxed{\langle P_{inj,i} \rangle = \dfrac{S_i}{2m_i}}$

iii) Mean power balance $\boxed{\langle P_{12} \rangle = B(\langle E \rangle - \langle E_2 \rangle) + o(\epsilon^2)}$

$$B = B_1 = B_2 = \frac{K^2(\Delta_1 + \Delta_2)}{m_1 m_2\left[(\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\Delta_1\omega_2^2 + \Delta_2\omega_1^2)\right]} \qquad B = \epsilon^2 B^0$$

The coupling factor *B* is a second order term

Subsystems of resonators



Assumptions

- conservative and light coupling
- subsystem = { uncoupled resonators + same m_i, c_i }
- « rain-on-the-roof » forces =

white noise forces + uncorellated forces + PSD constant in subsystems

subsystem *i, j* **mode** α, β

Canonical problem

$$\begin{cases} m_i\ddot{x}_{i\alpha} + c_i\dot{x}_{i\alpha} + k_{i\alpha}x_{i\alpha} = f_{i\alpha} + \sum_{j\neq i}\sum_\beta k_{i\alpha,j\beta}x_{j\beta} \\ E_{i\alpha} = \frac{1}{2}m_i\dot{x}_{i\alpha}^2 + \frac{1}{2}k_{i\alpha}x_{i\alpha}^2 - \sum_{j\neq i}\sum_\beta \frac{1}{2}k_{i\alpha,j\beta}x_{i\alpha}x_{j\beta} \\ P_{i\alpha,j\beta} = \frac{1}{2}k_{i\alpha,j\beta}(x_{i\alpha}\dot{x}_{j\beta} - \dot{x}_{i\alpha}x_{j\beta}) \end{cases}$$

$$E_i = \sum_\alpha E_{i\alpha}$$

$$P_{ij} = \sum_{\alpha,\beta} P_{i\alpha,j\beta}$$

By previous results $\langle P_{ij}\rangle = \sum_{\alpha,\beta}\epsilon^2 B^0_{i\alpha,j\beta}(\langle E_{i\alpha}\rangle - \langle E_{j\beta}\rangle) + o(\epsilon^2)$

But at order 0 ($\epsilon = 0$), $\langle E_{i\alpha}\rangle = \dfrac{S_i}{2c_i} + o(1)$

PSD constant

damping constant

Equipartition of energy

$$\langle E_{i\alpha}\rangle = \frac{\langle E_i\rangle}{N_i} + o(1)$$

Coupling power proportionality

$$\boxed{\langle P_{ij}\rangle = \left(\sum_{\alpha,\beta} B_{i\alpha,j\beta}\right)\left(\frac{\langle E_i\rangle}{N_i} - \frac{\langle E_j\rangle}{N_j}\right) + o(\epsilon^2)}$$

Random resonators

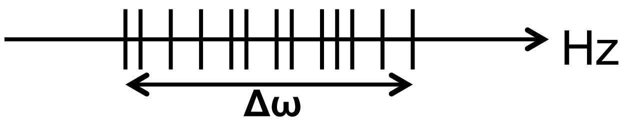
The coupling factor $\sum_{\alpha,\beta} B_{i\alpha,j\beta}$ is a very complicated function that depends on « blocked » natural frequencies and half-power bandwidth of all resonators. In practice, its effective computation is costly ($N \sim 10^6$) and requires the knowledge of all details of the subsystems.

This is exactly what we want to avoid in statistical energy analysis

To « forget » the exact position of eigenfrequencies, we need to approximate the discrete sum by an integral.

$$\frac{1}{N}\sum_{\alpha} f(\omega_\alpha) = \int f(\omega)p(\omega)d\omega \qquad \frac{1}{N_i N_j}\sum_{\alpha,\beta} f(\omega_\alpha,\omega_\beta) = \int\int f(\omega,\omega')p_i(\omega)p_j(\omega')d\omega d\omega'$$

How to choose the pdf?



Empirical cumulative distribution function

$$F_{\omega_i}(\omega) = \mathrm{Card}\left\{\alpha / \omega_{i\alpha} \le \omega\right\}$$

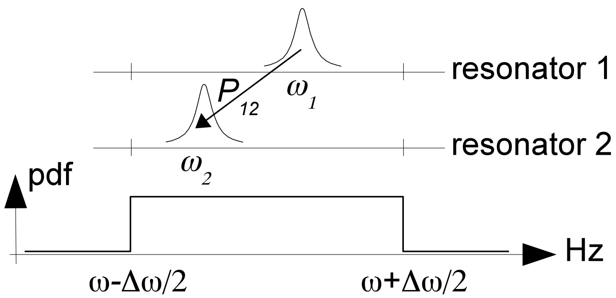
pdf

$$\boxed{p_i(\omega) = \frac{d}{d\omega}F_{\omega_i}(\omega) = \frac{n_i(\omega)}{N_i}}$$

modal density

# of modes in Δω

Uniform probability density function



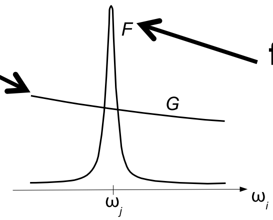
Assumptions

- light damping $\Delta_i << \Delta\omega$

half-power bandwidth **frequency bandwidth**

$$\sum_{\alpha,\beta} B_{i\alpha,j\beta} = \frac{K^2}{m_i m_j} \int_{\Delta\omega}\int_{\Delta\omega} \frac{(\Delta_i + \Delta_j)d\omega_i d\omega_j}{\left[(\omega_i+\omega_j)^2 + \frac{1}{4}(\Delta_i+\Delta_j)^2\right]\left[(\omega_i-\omega_j)^2 + \frac{1}{4}(\Delta_i+\Delta_j)^2\right]}$$

smooth term **fluctuating term**



$$\sum_{\alpha,\beta} B_{i\alpha,j\beta} = \frac{\pi K^2 N_i N_j}{2 m_i m_j \omega_0^2 \Delta\omega}$$

Coupling power proportionality

$$\langle P_{ij}\rangle = \omega_0 \eta_{ij} N_i \left(\frac{\langle E_i\rangle}{N_i} - \frac{\langle E_j\rangle}{N_i}\right)$$

coupling loss factor

Reciprocity

$$\eta_{ij} N_i = \eta_{ji} N_j$$

Ensemble of similar systems

Meaning of the approximation process

$$\frac{1}{N_i N_j} \sum_{\alpha,\beta} - \rightsquigarrow \int_{\Delta\omega} \int_{\Delta\omega} - d\omega d\omega'$$

System with a large number of modes $N_i >> 1$ and $N_j >> 1$

SEA applies to a unique system

Population of similar systems with random variations (Gibb's ensemble)

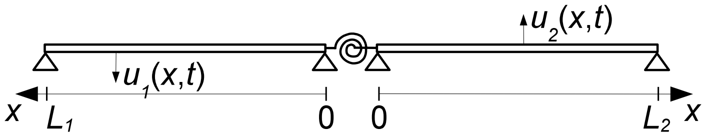
SEA applies to the average system

$$N_e \times N_m >> 1$$

number of members in the ensemble

number of modes in each member

Coupled beams   Lotz & Crandall 1971



Assumptions

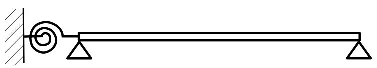
- non-resonant modes are neglected

Governing equation

$$m_i \frac{\partial^2 u_i}{\partial t^2} + c_i \frac{\partial u_i}{\partial t} + E_i I_i \frac{\partial^4 u_i}{\partial x^4} = f_i(x,t) + K \frac{\partial u_j}{\partial x} \delta'(x)$$

external force   **coupling force**

Blocked modes $\psi_{i\alpha}(x)$



- $E_i I_i \dfrac{d^4}{dx^4} \psi_{i\alpha}(x) = m_i \omega_{i\alpha}^2 \psi_{i\alpha}(x)$
- modes are orthogonal
- they form a complete set $u_i(x,t) = \sum_\alpha U_{i\alpha}(t)\psi_{i\alpha}(x)$

Reduction to the canonical problem

Subsituting $u_i(x,t) = \sum_\alpha U_{i\alpha}(t)\psi_{i\alpha}(x)$ into the governing equation leads to the following set of equations on modal amplitudes $U_{i\alpha}(t)$

$$
\begin{cases}
m_i \ddot{U}_{i\alpha} + c_i \dot{U}_{i\alpha} + m_i \omega_{i\alpha}^2 U_{i\alpha} = F_{i\alpha} + \sum_\beta K \psi'_{i\alpha}(0)\psi'_{j\beta}(0) U_{j\beta} \\
E_{i\alpha} = \frac{1}{2} m_i \dot{U}_{i\alpha}^2 + \frac{1}{2} m_i \omega_{i\alpha}^2 U_{i\alpha}^2 - \frac{1}{2}\sum_\beta K \psi'_{i\alpha}(0)\psi'_{j\beta}(0) U_{i\alpha} U_{j\beta} \\
P_{i\alpha,j\beta} = \frac{1}{2} K \psi'_{i\alpha}(0)\psi'_{j\beta}(0)(U_{i\alpha}\dot{U}_{j\beta} - \dot{U}_{i\alpha} U_{j\beta})
\end{cases}
$$

This is exactly the canonical problem

Coupling power proportionality

$$\boxed{\langle P_{ij} \rangle = \omega_0 \eta_{ij} n_i \left(\frac{\langle E_i \rangle}{n_i} - \frac{\langle E_j \rangle}{n_i} \right)}$$

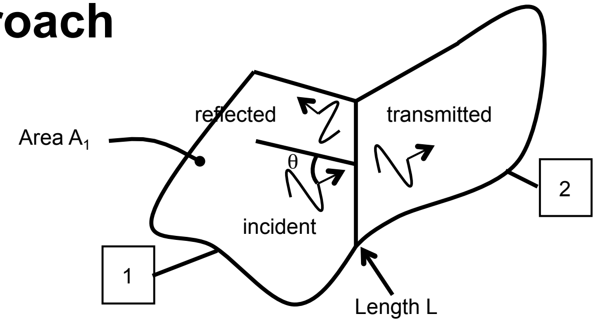
coupling loss factor **modal density**

$$\boxed{\eta_{ij} = \frac{K^2}{L_i (m_i E_i I_i)^{1/2} m_j^{1/4} (E_j I_j)^{3/4} \omega^{3/2}}}$$

length **mass per unit length** **bending stiffness**

Coupling loss factors

Wave approach



Assumptions

- diffuse field (homogeneous + isotropic)

$$P_{1\to2} = \omega\eta_{12}E_1$$

power transmitted from 1 to 2 **vibrational energy**

Specific intensity: $I_1 = c_{g_1}\dfrac{E_i}{2\pi A_i}$ Transmission efficiency : $T(\theta) = \dfrac{\mathcal{P}_{transmitted}}{\mathcal{P}_{incident}}$

$$P_{1\to2} = \int_L \int I_1 \cos\theta T(\theta) d\theta dL$$

$$P_{1\to2} = \frac{E_1}{2\pi A_1} c_{g_1} L \int_{-\pi/2}^{\pi/2} T(\theta)\cos\theta d\theta$$

Hence

Dimension 2

$$\boxed{\eta_{12} = \frac{Lc_{g_1}}{\pi\omega A_1}\int_0^{\pi/2} T(\theta)\cos\theta d\theta}$$

coupling length **plate area**

Dimension 3

$$\boxed{\eta_{12} = \frac{Sc_{g_1}}{4\pi\omega V_1}\int_0^{2\pi}\int_0^{\pi/2} T(\theta,\varphi)\cos\theta\sin\theta d\theta d\varphi}$$

coupling surface **room volume**

The problem reduces to the computation of the mean transmission efficiency

Identification procedure

System ={ n subsystems}

The subsystems are alternatively excited by a source of unit power.

E_j^i :energy in subsystem j for a unit power in subsystem i

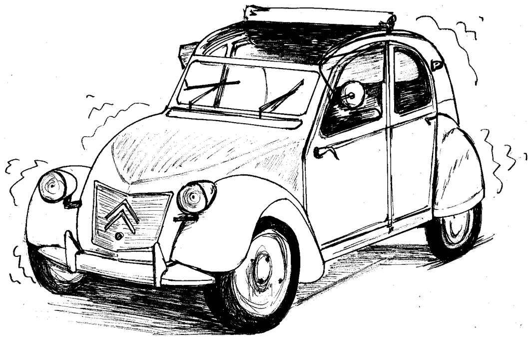
$$\underset{\textbf{imposed}}{\begin{pmatrix} 0 \\ \vdots \\ P_i = 1 \\ \vdots \\ 0 \end{pmatrix}} = \underset{\textbf{unknown}}{\begin{pmatrix} \eta_{11} & \cdots & \eta_{1n} \\ \vdots & & \vdots \\ \eta_{n1} & \cdots & \eta_{nn} \end{pmatrix}} \underset{\textbf{measured}}{\begin{pmatrix} E_1 \\ \vdots \\ \vdots \\ E_n \end{pmatrix}} \qquad n \text{ equations}$$

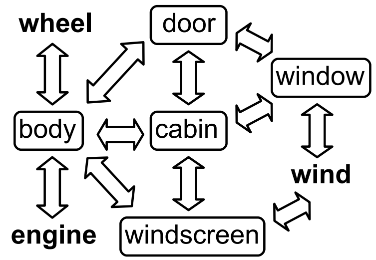
Repeating the experiment for other source subsystems gives n^2 equations. The coupling loss factors   are obtained by solving these equations.

Remark : reciprocity of CLF has not been used.

It exists many variants of this procedure

Energy balance





For each subsystem, the energy balance reads $$\underbrace{P_{inj,i}}_{\text{injected power}} = \underbrace{P_{diss,i}}_{\text{dissipation}} + \underbrace{\sum_{j \neq i} P_{ij}}_{\text{exchanged power}}$$

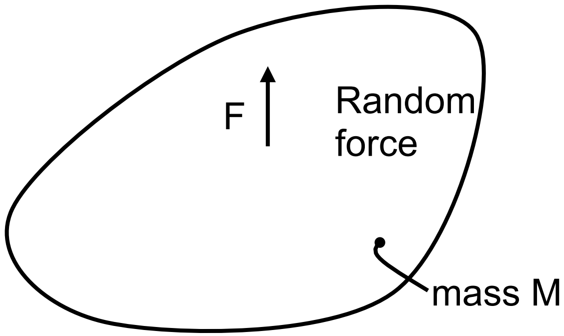
Loss by damping $P_{diss,i} = \eta_i \omega E_i$

Loss by coupling $P_{ij} = \omega \eta_{ij} E_i - \omega \eta_{ji} E_j$

Hence $$P_i = \eta_i \omega E_i + \sum_{j \neq i} (\omega \eta_{ij} E_i - \omega \eta_{ji} E_j)$$

In a matrix form $$\boxed{\omega \begin{pmatrix} \sum_j \eta_{1j} & & -\eta_{ji} \\ & \ddots & \\ -\eta_{ji} & & \sum_j \eta_{nj} \end{pmatrix} \begin{pmatrix} E_1 \\ \vdots \\ E_n \end{pmatrix} = \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix}}$$

Injected power



Injected power $$P = \langle f v \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{fv}(\omega)d\omega$$

force **velocity**

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ff}(\omega)Y(\omega)d\omega$$

mobility

$$P = \langle f^2 \rangle \frac{1}{\Delta\omega}\int_{\Delta\omega} \Re\left[Y(\omega)\right] d\omega$$

A statistical estimation of the mean conductance (real part of the mobility) gives,

injected power **modal density**

$$\boxed{P = \langle f^2 \rangle \times \frac{\pi n(\omega)}{2M}}$$

square mean force **mass of subsystem**

Entropy in statistical analysis

Clausius (1822-1888)

<u>Clausius' postulate</u>

Heat cannot be transferred from cold body to hot body without converting heat into work.

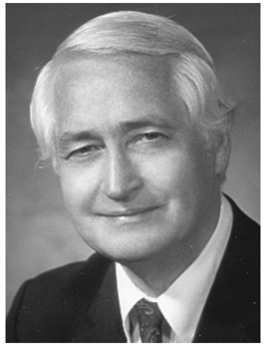
$$\boxed{T_2 > T_1} \xrightarrow{Q} \boxed{T_1}$$

Heat spontaneously flows from hot body to cold body

<u>Coupling power proportionality</u>

$$P_{ij} = \omega \eta_{ij} N_i \left(\frac{E_i}{N_i} - \frac{E_j}{N_j} \right)$$

Vibrational power **High modal energy** **Low modal energy**



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Vibrational energy flows from high modal energy to low modal energy

Analogy : Heat δQ $\leftrightarrow$ Vibrational energy dE

Temperature T $\leftrightarrow$ Modal energy **E/N**

Clausius entropy : $dS = \dfrac{\delta Q}{T}$ $\leftrightarrow$ Vibrational entropy

Variation of energy

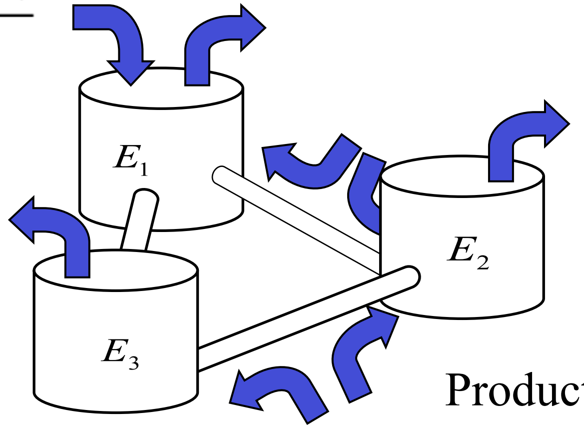
$$\boxed{dS = N \frac{dE}{E}}$$

Vibrational entropy **# of modes**

Entropy balance

Production of entropy

$$\frac{dS_i^{\mathrm{inj}}}{dt} = k\frac{P_i^{\mathrm{inj}} N_i}{E_i}$$



Dissipation of entropy

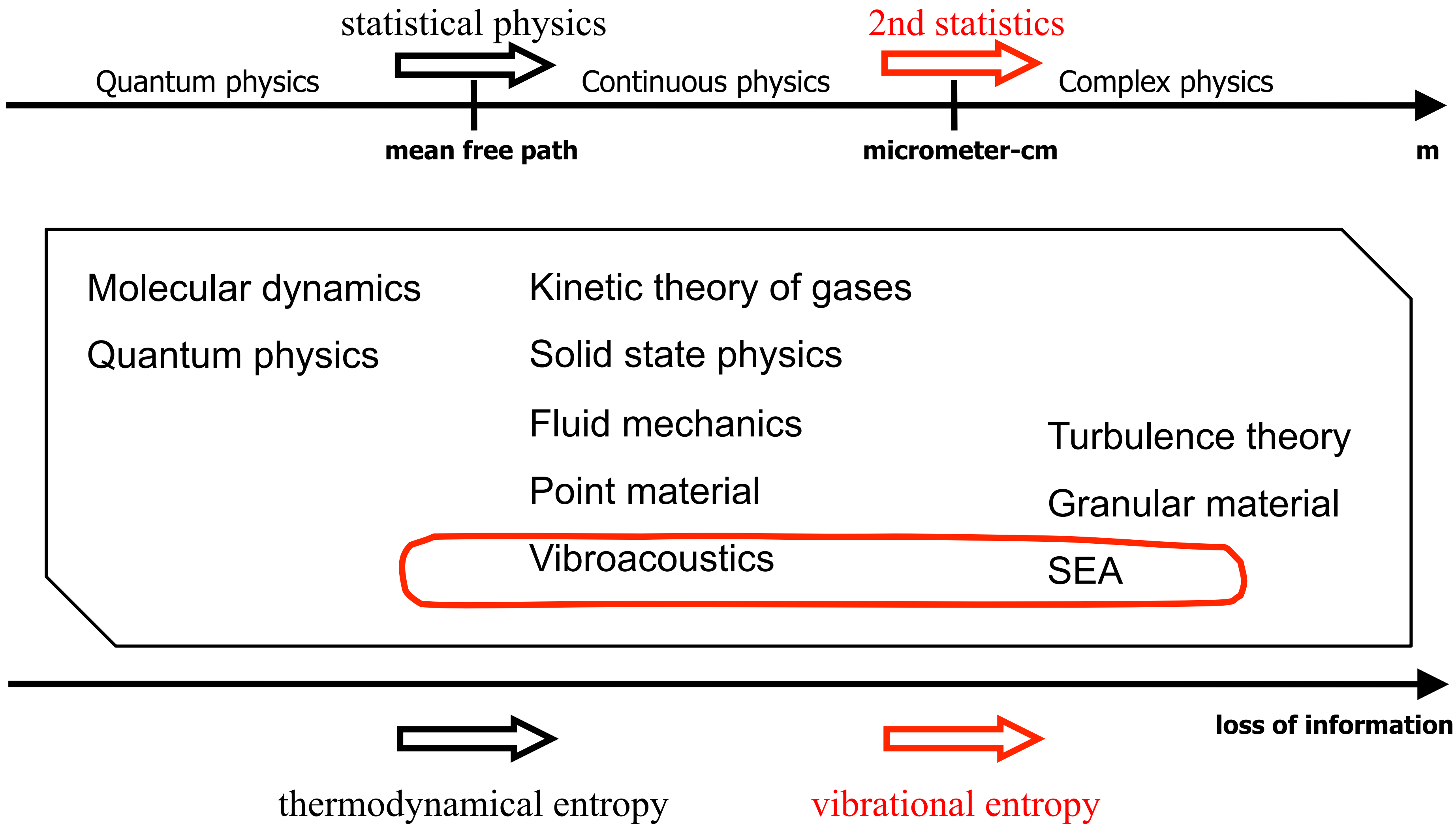
$$\frac{dS_i^{\mathrm{diss}}}{dt} = -k\omega\eta_i N_i$$

Production of entropy by mixing

$$\frac{dS_{ij}}{dt} = k\omega(\eta_{ij}E_i - \eta_{ji}E_j)\left(\frac{N_j}{E_j} - \frac{N_i}{E_i}\right)$$

$$\boxed{\frac{dS}{dt} = 0}$$

Statistics at large scale



Vibrational entropy is a measure of missing information between FEM and SEA.