Statistical energy analysis made simple, and difficulties with strong coupling

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Statistical Energy Analysis

• “SEA: Seems Easy, Aint”
  – (anon)

• “You must have faith” .. when using SEA
  – (Bob Craik)

• Don’t be intimidated
  – You have reasons to use SEA
  – SEA is difficult but no more so than, say, the FEM
Stockholm Concert Hall

- Play Cello on Floating Floor
  - Very trendy
  - Endpin induces floor vibrations
    - Radiated sound increase loudness
    - Risk Low Frequency Absorption
    - Too high already

https://open.spotify.com/track/3Mle6ILTYnFdNxb60USbgQ

\[
\sigma = \sigma \left( \frac{k_a}{k_s}, k_a^2 S, k_a P \right)
\]

\(k_a\) – Fluid wave number; \(k_s\) – Structure wave number
Acoustic Fatigue in Rocket-to-the-Moon

- The origin of SEA
- Some 200,000 modes in structure
- Acoustic field not quite known but it is powerful

“The principle of vibroacoustic reciprocity”
- Relates sound radiation and sound reception

P. W. Smith 1962 JASA 34, 640-647. Response and radiation of structural modes excited by sound:

\[ M_s \hat{e}_s + C \left( \hat{e}_s - \hat{e}_a \right) = 0 \]

\[ C = \frac{\rho_o c n_i}{\mu_s} \sigma; \quad M_s = (\eta \omega n)_s; \quad n_s = \left( \frac{k S}{2\pi c_s} \right)_s \]

\[ \hat{e}_i = \frac{E_i}{n_i}; \quad \hat{e}_a = \left( \frac{2\pi^2 c}{\rho_0 \omega^2} \right)_a \left\langle \hat{p}_a^2 \right\rangle; \quad \hat{e}_s = \left( \frac{2\pi c_s \mu}{k} \right)_s \left\langle \hat{v}_s^2 \right\rangle \]

\[ \eta_s ? k_s (\omega) ? \]
Simply Supported Plate

$F$

$0.7 \text{ m}$

$1.15 \text{ m}$

$h = 1 \text{ mm}$

Input Power (W)

Frequency (Hz)
Simply Supported Plate ..

\[ F \]

1.15 m

0.7 m

\[ h = 1 \text{ mm} + \sigma \]
\[ \sigma = N(0, 20 \mu \text{m}) \]

Acoustic limit:
\[ \sigma_\omega \approx \delta \omega \]

91 Hz

100 realisations

\[ M \approx 1 \text{ at } 400 \text{ Hz} \]
Simply Supported Plate ...
"Magic Integral"

\[
\langle P_{in} \rangle_{x_o, \Delta \omega} = \frac{1}{\Delta \omega} \text{Re} \int_{\Delta \omega} \sum_r \frac{-i \omega \langle |F \phi_r(x_0)|^2 \rangle_{x_0}}{m_r \left( \omega_r^2 - 2i \omega \omega_r \xi_r - \omega^2 \right)} d \omega
\]

\[
\approx \left\langle \frac{|F_r|^2}{m_r} \right\rangle_r \frac{1}{\Delta \omega} \int_{\Delta \omega} \sum_r \frac{\pi}{2} \delta(\omega - \omega_r) d \omega \approx \frac{\pi |F|^2}{2 m} \frac{\Delta N}{\Delta \omega}
\]
\[ I = \int_{w_t}^{w_u} \Re \left( \frac{i \omega}{(w_r^2 - \omega^2 + i \omega w_r \rho)} \right) d\omega \]

\[ = \int_{w_t}^{w_u} \frac{w^2 w_r \rho}{(w_r^2 - \omega^2)^2 + (\rho \omega w_r)^2} d\omega \]

Assume

\[ i) \quad \frac{w_r - w_t}{\rho w_r} \gg 1 \]

\[ ii) \quad \frac{w_u - w_r}{\rho w_r} \gg 1 \]

Most input power is in the frequency band

\[ \omega_r \approx \omega, \text{ at frequencies for which the integrand is large} \]
\[ I \approx \frac{1}{\gamma} \sum_{\omega_e} \frac{\eta \omega_r}{(\omega_r - \omega)^2 + (\omega_r \eta/2)^2} \]

\[ = \frac{1}{2} \left[ Q \tan \left( \frac{\omega - \omega_r}{\eta \omega_r} \right) \right] \omega_u \]

\[ \approx \frac{1}{2} \left[ Q \tan (\pm \infty) - Q \tan (-\infty) \right] \]

\[ = \frac{\pi}{2} \tan^{-1} \frac{1}{2} \quad \Rightarrow \langle P_{in} \rangle_{x_0, \Delta \omega} \approx \frac{k_s |F|^2}{4 \mu_s c_g} \]

Input Power independent of: \textit{i}) size, and \textit{ii}) damping
Very Large Homogenous structure, excited at a Random Location.

\[ P_{in} = \text{Re} \left( \sum_n \frac{-i \omega |F_n|^2}{m_n (\tilde{\omega}_n^2 - \tilde{\omega}^2)} \right) \]

\[ \approx \frac{|F_0|^2}{m} \text{Re} \left( \sum_n \frac{-i \omega}{(\tilde{\omega}_n^2 - \tilde{\omega}^2)} \right) = \frac{|F_0|^2}{m} \delta \omega_n \text{Re} \left( \sum_n \frac{-i \omega}{(\tilde{\omega}_n^2 - \tilde{\omega}^2)} \delta \omega_n \right) \]

\[ \approx \frac{|F_0|^2}{m} \frac{1}{\delta \omega} \text{Re} \left( \int \frac{-i \omega}{(\tilde{\omega}_n^2 - \tilde{\omega}^2)} \text{d}\omega_n \right) \]

\[ = \frac{\pi n}{2 m} |F_0|^2 \]

Riemann Sum,

Valid approximation if M > 1
Random structure

Random eigenfrequencies. Probability density: $p(\omega_n)$

$$P_{in} = \int \sum_n p(\omega_n) \text{Re} \left( \frac{-i \omega |F_n|^2}{m_n (\tilde{\omega}_n^2 - \tilde{\omega}^2)} \right) \, d\omega_n$$

Rectangle distributed: $p(\omega_n) = \begin{cases} 
\frac{1}{\omega_u - \omega_l}, & \omega_l < \omega_n < \omega_u \\
0 & \text{otherwise}
\end{cases}$

$$P_{in} = \sum_n \frac{1}{\omega_u - \omega_l} \int_{\omega_l}^{\omega_u} \text{Re} \left( \frac{-i \omega |F_n|^2}{m_n (\tilde{\omega}_n^2 - \tilde{\omega}^2)} \right) \, d\omega_n \approx \frac{\pi}{2} \left\langle \frac{|F_n|^2}{m_n} \right\rangle_n \frac{\Delta N}{\Delta \omega}$$

One substructure: ergodicity
Power Balance:

\[ P_{in.i} = P_{d.i} + \sum_{j \neq i} P_{c.i.j} \]
“Compact Support”

Conductivity given by:

\[ P_{c.1.3} = C^{1.3} (\hat{e}_1 - \hat{e}_3) \]
One – Way Estimate

Conductivity given by:

\[ C^{1.3} = \left( \frac{P_{c.1.3}}{\hat{e}_1} \right) \]

i.e., we use this Eq. to calculate the conductivity
SEA Formulation ...

\[ P_{d,i} = \eta_i \omega E_i = M_i \hat{e}_i \]  
Linear proportional damping

\[ P_{in} \]  
Independent of Connected Elements

\[ P_{coup}^{i,j} = C^{i,j} (\hat{e}_i - \hat{e}_j) \]  
Coupling Power Proportionality (CPP)

\[ \hat{e}_i = E_i / n_i \]  
Modal Power

\[ n \]  
Modal Density

\[ M = \eta \omega n \]  
Modal Overlap Factor

\[ C^{i,j} = \eta_{coup}^{i,j} \omega n_i \]  
Conductivity
Yuet-Yan Pang
“Air-Borne Sound Transmission through Extruded Profiles”

- Air-borne sound transmission from bogie to interior – medium and high frequencies.
2-D FE – Model

\[ U(x, y, z, t) = \text{Re} \left( \left[ \Psi(x, y) \right]^T V(z) e^{-i\omega t} \right) \]

\( \Psi \) – FE - Shape Functions

\( V \) – Nodal Displacements
Statistical Energy Analysis
Low Frequency Model

\[
\begin{align*}
W_{31} &= \omega \eta_{31} E_3 \\
W_{12} &= \omega \eta_{12} E_1 \\
W_{13} &= \omega \eta_{13} E_1 \\
W_{23} &= \omega \eta_{23} E_2 \\
W_{32} &= \omega \eta_{32} E_3 \\
W_{1d} &= \omega \eta_{1d} E_1 \\
W_{2d} &= \omega \eta_{2d} E_2 \\
W_{3d} &= \omega \eta_{3d} E_3
\end{align*}
\]
Two AutoSEA Models

An orthotropic plate – Many Plates
Plates line-coupled along a beam

RS Langley, KH Heron JSV 1990
“Elastic wave transmission through beam/plate junctions”

Diffuse Field in Element 1:

\[ \tau(\phi) = \frac{W_{ir}(\phi)}{W_{in}(\phi)} \]

\[ C = \frac{k_1 L}{4\pi} \left\langle \tau(\phi) \right\rangle_{\phi} \]
Transmission Loss Result for Metro train floor

Method 1, Rw = 28 dB
Measured curve, Rw = 28 dB
Mass law, Rw = 43 dB
Method 2, Rw = 27 dB
Transmission Loss Result for Regional train

<table>
<thead>
<tr>
<th>Frequency [Hz]</th>
<th>Transmission Loss [dB]</th>
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<tbody>
<tr>
<td>10</td>
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<td>60</td>
<td>10</td>
</tr>
<tr>
<td>70</td>
<td>10</td>
</tr>
</tbody>
</table>

- Method 1, Rw = 26 dB
- Measurements made by Kohrs Rw = 27 dB
- Method 2 Rw = 26 dB
- Masslaw, Rw = 46 dB
KTH-Enable Experiments

Anechoic Room
Reverberation Room
Shock and Vibration Room
Semi-Anechoic Room

\[ V_{\text{max}} = 130 \text{ m/s} \]

Background Noise < 25 dB
Measurement Setup

- Plate vibration caused by
  - Turbulence?
  - Tunnel vibration?
Vibrations of Plate and Tunnel

Sufficient difference?

Fig. 15. Accelerations with 120 m/s flow speed in original wind tunnel. —, tunnel; --, test plate.
Example: Wind tunnel

\( V_P \) caused by \( P \) or \( V_T \)?

1. Tunnel only is excited, find

\[
H_T \equiv \left( \frac{\langle V_P^2 \rangle}{\langle V_T^2 \rangle} \right)_T
\]

2. Under operation, measure

\[
H_M \equiv \left( \frac{\langle V_P^2 \rangle}{\langle V_T^2 \rangle} \right)_M
\]

3. If \( H_T \ll H_M \) OK

\( H_T ? \)
SEA, tunnel excited

\[
\begin{bmatrix}
M_T + \varepsilon & -\varepsilon \\
-\varepsilon & M_p + \varepsilon \\
\end{bmatrix}
\begin{bmatrix}
\hat{e}_T \\
\hat{e}_p \\
\end{bmatrix}
= 
\begin{bmatrix}
p_{in} \\
0 \\
\end{bmatrix}
\]

Whatever the values of \(M_T, M_p\) and \(\varepsilon\):

\[
\hat{e}_p < \hat{e}_T
\]

\[
m_p \frac{\left\langle \vec{v}_p^2 \right\rangle}{n_p} < m_T \frac{\left\langle \vec{v}_T^2 \right\rangle}{n_T}
\]

\[
m_p = s_p t_p s_p, \quad m_T = s_T t_T s_T
\]

**Conservative Estimate of** \(H_T\)

\[
H_T = \frac{\left\langle \vec{v}_p^2 \right\rangle_T}{\left\langle \vec{v}_T^2 \right\rangle_T} \leq \frac{m_T}{m_p} \cdot \frac{n_p}{n_T}
\]

If \(H_M \gg \frac{m_T}{m_p} \cdot \frac{n_p}{n_T} > H_T\) then OK

Need modal densities
Thin-walled plate:

\[
\frac{m_p}{n_p} = \frac{S_p \sigma \varepsilon_p}{S_p / (3.6 \varepsilon_p \rho)} = 3.6 \left( \frac{\sigma \varepsilon^2}{\rho} \right)
\]

Tunnel: 12 mm steel; Plate: 1.6 mm Aluminium

At lower frequencies, tunnel as a beam

At higher frequencies, tunnel as a plate assembly

\[
\frac{m_T}{n_T} = 3.6 \left( \frac{\sigma \varepsilon^2}{\rho} \right)
\]

\[
10 \log \left( H_T \right) < 10 \log \left( \frac{(\sigma \varepsilon^2)_T}{(\sigma \varepsilon^2)_P} \right) = 22 \text{ dB}
\]

High/Low Frequencies

Other Frequencies

\[ \approx \]
Waves in Tunnel

For each wave, \( r \):

\[
n_r = \frac{L}{\pi c_{g,r}} = \frac{L}{\pi} \frac{\partial k_r}{\partial \omega}
\]

\[
\left( \frac{n}{m} \right)_T = \frac{1}{\pi (\rho S)_T} \sum_r \frac{\Delta k_r}{\Delta \omega}
\]
Final Result

- Viscoelastic damping on tunnel (improves at high frequencies)
- Blocking masses (solves 300 Hz problem)

Thus:

$$H_M \gg H_T$$
**Conclusions**

- SEA elements are elements of response NOT elements of substructures
- SEA is built upon assertions of the vibroacoustic field in classes of structures:
  - Diffuse field in a room
  - Reverberant motion of plate with uncertain properties
  - ..
- SEA software are libraries for such “templates”
  - Trick is to know when and how to use them
    - Diagnostic measurements
    - Alternative calculations
    - Experience

- Wind tunnel: Just saying “SEA works” -> conservative upper bound
- Railway car: Once the templates are identified, the rest is easy
- Concert hall: Given $\eta$ and $k$ for floor, it’s easy to estimate
  - Sound radiation
  - Low frequency sound absorption

- SEA is very useful
- “Standard” SEA is built upon One-Way procedures for CLFFs
One-Way procedures for CLFs

- One junction at a time (compact support)
- Define field in first element
  - Level \( \hat{\epsilon} \), “diffuse wave field”, “resonant modes”, ...
- Express \( P_{tr} \)
  - Possibly, for “Weak Coupling”
  - Possibly, for infinite receiving element
  - Possibly, for element with random properties
- Express coupling loss
  \[ C_{1,2} = \omega n_1 \eta_c^{(1,2)} = P_{tr}^{(1,2)} / \hat{\epsilon}_1 \]
Examples of One-Way procedures for CLFs

- ISO standards for Sound Reduction Index ensures the conditions for One-Way procedures
- Sound Reception (Smith 1962)
- Sound radiation (Maidanik 1962, Leppington 1982)
- Walls and Double walls (Price 1970, Craik 2003, Finnveden 2007)
- Point coupling (Lyon 1975)
- Structural line coupling (Gibbs 1974, Langley 1990)
- ...
- Shorter & Langley (2005) General Smith theory
- Le Bot (2007) Radiative exchanges

All of these: "Vibroacoustic Reciprocity"

\[ C_{1,2} = C_{2,1} \]
Example: Cello exciting floor

\[ (k(1+i\eta)-m\omega^2)\ddot{u} = -\ddot{F} \]

\[ F = Z_{plate}i\omega\ddot{u} \]

\[ m\left(\omega_o^2 - \omega^2 - i\omega\text{Im}(Z_{plate}/m)\right)\ddot{u} + i\omega_o^2m(\eta + \eta_c)\ddot{u} = 0 \]

\[ \eta_c = \omega\text{Re}(Z_p)/\omega_o^2m \]

One mode: \( n = \frac{1}{\Delta\omega} \); \( \Delta\omega \) – Analysis band width

Conductivity: \( C = \omega n \eta_c \approx \text{Re}(Z_p)/\Delta\omega m \)

CLF is here for a rigid connector. It’s based on the assumption of “weak coupling”
One-Way procedures for CLFs...

• One junction at a time (compact support)
  – FRFs defined by elliptic equations, so, this cannot be exact
  – Doesn’t work for 3 coupled oscillators (Woodhouse 1989)
  – Exact CLFs depend on damping, thus they depend on coupling
damping at the next junction (Finnveden 1995)

• Define field in first element
  – Element has Finite Impedance / Finite Mobility, so, it cannot be
  an energy source

• SEA requires Weak Coupling
  – Errors when it’s not Weak Coupling?
    • Errors in dB?
    • Errors in Physics -> errors in trends?
  – When is it strong coupling?
  – What can we do?
Dynamic Coupling Strength / Strength of Connection

Vibroacoustic Reciprocity:  \( C_{1,2} = C_{2,1} \)

\[
\left( \tilde{v}_1^2 \right)_{\text{end}} \approx \left( e_1 \right)_{\text{end}} \approx \frac{\hat{e}_1 n_1}{L_1 \mu_1} \approx \frac{\hat{e}_1}{\pi (\mu c_L)_1} \\
P_c^{(1,2)} = \text{Re} \left( Z_2 \right) \left( \tilde{v}_1^2 \right)_{\text{end}} = \frac{1}{\xi_2} \frac{\hat{e}_1}{\pi} \\
\left( \tilde{f}_1^2 \right)_{\text{end}} \approx (EA)_1 \left( e_1 \right)_{\text{end}} \approx (\mu c_L)_1 \frac{\hat{e}_1}{\pi} \\
P_c^{(1,2)} = \text{Re} \left( Y_2 \right) \left( \tilde{f}_1^2 \right)_{\text{end}} = \xi_2 \frac{\hat{e}_1}{\pi} \\
P_c^{(1,2)} = \text{Re} \left( Z_2 \right) \left( \tilde{v}_1^2 \right)_{\text{end}} = \frac{2}{\pi} \frac{\text{Re} \left( \xi_2 \right)}{1 + \xi_2^2} \frac{\hat{e}_1}{\pi} \\
n_i = L_i / \pi c_{g,i} \quad \xi_2 = \frac{\left( \mu c_L \right)_2}{\left( \mu c_L \right)_1}
Very strong coupling

\[ C \gg M_i, \quad C \gg M_j \Rightarrow \hat{e}_1 \approx \hat{e}_2 \]

"Smith's strong coupling criterion"

“SEA gives the right answer for the wrong reason”

A.J. Keane

\[
\begin{bmatrix}
M_1 + C^{1,2} + C^{1,3} & -C^{1,2} & -C^{1,3} \\
-C^{1,2} & M_2 + C^{1,2} + C^{2,3} & -C^{2,3} \\
-C^{1,3} & -C^{2,3} & M_3 + C^{2,3} + C^{1,3}
\end{bmatrix}
\begin{bmatrix}
\hat{e}_1 \\
\hat{e}_2 \\
\hat{e}_3
\end{bmatrix}
= 
\begin{bmatrix}
\Pi_m \\
0 \\
0
\end{bmatrix}
\]

- Tunnelling is a mathematical artefact
  - When global modes dominates response
  - When spatial damping decay matters
- Barbagallo ISMA 2010: \( \eta L \omega / c_g > 1 \)

What is the Right Reason?
Impulse Response

\[ M_1 \ddot{x}_1 + C_1 \dot{x}_1 + (K_1 + K_c)x_1 - K_c x_2 = 0 \]

\[ M_2 \ddot{x}_2 + C_2 \dot{x}_2 + (K_2 + K_c)x_2 - K_c x_1 = 0 \]

\[ x_1 = x_2 = \dot{x}_2 = 0, \quad \dot{x}_1 = 1/m_1 \text{ at } t = 0 \]

\[ x_2 = A_1 \left( \frac{1}{b_1} e^{-\alpha_1 t} \sin \Omega_1 t - \frac{1}{b_2} e^{-\alpha_2 t} \sin \Omega_2 t \right) \]

\[ \alpha_{1,2} = -\text{Im} \sqrt{\tilde{\omega}_s^2 \pm \sqrt{(\tilde{\omega}_1^2 - \tilde{\omega}_2^2)^2 / 4 + \chi^2}} \]

\[ \Omega_{1,2} = \text{Re} \sqrt{\tilde{\omega}_s^2 \pm \sqrt{(\tilde{\omega}_1^2 - \tilde{\omega}_2^2)^2 / 4 + \chi^2}} \]

\[ \tilde{\omega}_s = \sqrt{(\tilde{\omega}_1^2 + \tilde{\omega}_2^2) / 2} \]

\[ \tilde{\omega}_1^2 = \left( k_i + k_c \right) / m_i = \omega_i^2 (1 - i \eta_i) \]

\[ \chi = k_c / \sqrt{m_1 m_2} \]

Wearing pushes coupled frequencies apart
Response of Oscillator 2
Response of Oscillator 2 ..

See also Lyon&DeJong pp 52 (equal oscillators)

Failed to do a decent parallel to Einstein (1905): Viscosity from Brown’s motion ... (fantastic!)
Kinetic Energy, 2\textsuperscript{nd} Oscillator

\[ e_k = 2 \, e^{-T} \sin^2 \kappa T \]

\[ T = \eta \omega t; \]

\[ \kappa = 2 \sqrt{\delta + \gamma} \]

\[ \delta = \left( \frac{\omega_1 - \omega_2}{\eta \omega} \right)^2 \]

\[ \gamma = \frac{k_c^2}{\eta^2 \omega^4 \, m_1 \, m_2} \]

\[ \eta = \eta_1 = \eta_2 \]

Fahy & James (JSV -96) measured this for coupled plates. Response in plate 2 is dominated by mode-pairs: \( \delta < 1 \)

Across an ensemble

\[ \gamma < 1 \Rightarrow \kappa < 1 \]
Steady State Energy Flow

\[
(k_1 + k_c - i \omega c_1 - \omega^2 m_1) U_1 - k_c U_2 = F_1,
\]

\[
-k_c U_1 + (k_2 + k_c - i \omega c_2 - \omega^2 m_2) U_2 = 0
\]

\[
P_{m.1} = \text{Re}\left(-i \omega u_1 f^*\right) = \frac{(r_2^2 + 1) + \gamma}{(r_1 r_2 - \gamma - 1)^2 + (r_1 + r_2)^2} \frac{\omega |f|^2}{m_1 \Delta_1},
\]

\[
P_{coup}^{1.2} = \text{Re}\left(-i \omega u_1 k_c (u_1 - u_2)^*\right) = \frac{\gamma}{(r_1 r_2 - \gamma - 1)^2 + (r_1 + r_2)^2} \frac{\omega |f|^2}{m_1 \Delta_1}
\]

\[
r_i = \frac{(\omega_i^2 - \omega^2)}{(\eta \omega_i^2)}
\]

\[
\omega_i^2 = \frac{(k_i + k_c)}{m_i}
\]

\[
\gamma = \frac{k_c^2}{\eta_1 \eta_2 \omega^4 m_1 m_2}
\]
Figure 1. Normalised coupling power, $P_{\text{coup}}/|\omega|^2/m_1 \Delta_i$, for $\gamma = 0.1$. The non-

$$r_i = \left( \omega_i^2 - \omega^2 \right) / (\eta_i \omega^2) \approx 2 \left( \omega_i - \omega \right) / (\eta_i \omega)$$
\[ \gamma = 10 \]

Figure 2. Normalised coupling power, \( \frac{P_{\text{coup}}^{1/2}}{\omega |f|^2/m_i \Delta_i} \), for \( \gamma = 10 \).
Normalised Coupling Power

\[ \gamma = 0.02 \]

\[ \gamma = 0.5 \]

\[ \gamma = 2 \]

\[ \gamma = 50 \]

\( \gamma \) - "Modal Interaction Strength"

Wearing is not apparent in response if: \( \gamma < 1 \)
Input Power (Input Mobility)

Langley – 89: “Coupling is weak if input mobility is unaffected by connected elements”  true if: $\gamma < 1$
One-Way Approximation (standard SEA) ..

Motion of first Osc is a given quantity

\[ \Rightarrow \langle \hat{e}_1 \rangle = \omega^2 M_1 \tilde{u}_1^2 / n_i \]

\[ \Rightarrow \langle \hat{e}_2 \rangle \Rightarrow \langle P_{\text{dissipated}} \rangle = \langle P_{\text{coup}} \rangle \]

\[ \langle P_{\text{coup}} \rangle = C_e \langle \hat{e}_1 \rangle \]

1) One resonance in band \( \Delta \omega = \omega_u - \omega_l \)

\[ \Rightarrow n_i = 1 / \Delta \omega \]

2) Replace \( \omega \approx \omega_1 \approx \omega_2 \), wherever possible

3) Use magic integral

\[ C_e = \frac{\pi}{2} \frac{|\chi(\omega)|^2}{\omega^2 (\Delta \omega)^2} \]

\[ \chi = \frac{(k_c + i \omega g_c - \omega^2 m_c / 4)}{\sqrt{m_1 m_2}} \]
Scharton & Lyon JASA (1968)

One, out of very few, demonstrations of CPP based on fully coupled solutions

\[ \langle P_c^{(1,2)} \rangle_\omega = \beta \left( \langle E_{k,1} \rangle_\omega - \langle E_{k,2} \rangle_\omega \right) \]

\[ \mathcal{B} = \left\{ \mu^2 \left[ \Delta_1 \omega_2^4 + \Delta_2 \omega_1^4 + \Delta_1 \Delta_2 (\Delta_1 \omega_2^2 + \Delta_2 \omega_1^2) \right] \right. \]

\[ \left. + (\gamma^2 + 2\mu \kappa)(\Delta_1 \omega_2^2 + \Delta_2 \omega_1^2) + \kappa^2 (\Delta_1 + \Delta_2) \right\} \]

\[ \times \left\{ (1 - \mu^2) \left[ (\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\Delta_1 \omega_2^2 + \Delta_2 \omega_1^2) \right] \right\}^{-1} \]

\[ \Delta_i = R_i / M_i' ; \quad \omega_i^2 = (K_i + K_c) / M_i' ; \quad M_i' = M_i + M_c / 4 ; \]

\[ \kappa^2 = K_c^2 / M_1' M_2' ; \quad \gamma = G M_1' M_2' ; \quad \mu = M_c \lambda / 4 ; \quad \lambda = M_1' / M_2' \]
Frequency averaged response of Two Coupled Oscillators

- The Power flow is proportional to the difference of the Oscillators’ Energy
- The constant of proportionality is positive definite
- The constant of proportionality is Symmetric in system parameters
  - $\rightarrow$ CPP

- If one oscillator is excited, the energy of the second oscillator cannot be greater than that of the first oscillator
- However, the constant of proportionality depends on oscillator damping.
  - Hence, CPP cannot be exact for three oscillators
    - As been proven (Woodhouse 1981)
Ensemble Averages

• Lyon 1975, Mace and Ji 2007

\((\omega_1 - \omega_2)\) is rectangle distributed.

\[
\text{pdf}(\omega_1 - \omega_2) = \begin{cases} 1/\Delta\omega & \text{in band} \\ 0 & \text{otherwise} \end{cases}
\]

\(\Delta\omega\) is wide enough

Replace \(\omega \approx \omega_1 \approx \omega_2\), wherever possible

\[
C = C_e/\sqrt{1 + \gamma};
\]

\[
\gamma = \frac{2}{\pi} \frac{C_e}{M_1 M_2} = \frac{2}{\pi} \frac{\pi}{2} \frac{\chi(\omega_n)^2}{\eta_1 \eta_2 \omega^4} = \frac{\chi(\omega_n)^2}{\eta_1 \eta_2 \omega^4}
\]

If \(\gamma < 1\), the one-way approach is OK, otherwise the Connectivity depends on damping (and thus on coupling damping)
Reynders 2014

- Weaker losses -> Stronger modal interaction strength
  - SEA over predicts transmission
  - But not much

Upper curves $\eta = 0.01$; Lower curves $\eta = 0.001$

Seems as the values of $\gamma$ is incorrect in the article

Black dash-dot: 'SEA'
SEA of Two-element Structure
(One more, out of very very few, fully coupled demonstrations)

\[ M_i = \frac{\eta_i k_0 L_i}{\pi} \]
\[ C_e = \frac{2 \text{ Re}(\zeta_2)}{\pi \left(1 + |\zeta_2|^2\right)} \]

Figure 3-9. Two-element structure.
Energies and Energy Flows

\[ e_p = \frac{S_1 |\tilde{p}_1|^2}{2 \rho_0 c} = \frac{\Pi_{in}}{2 c} \frac{\sin(kx + \psi)}{\cos(kL - \psi)}^2, \]

\[ e_k = \frac{S_1 \rho_0 |\tilde{\nu}_1|^2}{2} = \frac{\Pi_{in}}{2 c} \frac{\cos(kx + \psi)}{\cos(kL - \psi)}^2, \]

\[ \Pi_{in} = \rho_0 c |\tilde{\nu}_0|^2 S_0^2 / S_1. \]

\[ P_{in} = S_1 \text{Re} \left( \tilde{p}_1 (x = -L) \tilde{\nu}^*_1 (x = -L) \right) = \Pi_{in} \text{Re} \left( -i \tan(kL - \psi) \right), \]

\[ P_{tr} = S_1 \text{Re} \left( \tilde{p}_1 (x = 0) \tilde{\nu}^*_1 (x = 0) \right) = \Pi_{in} \frac{\sinh(-2 \text{Im}(\psi))}{2 |\cos(kL - \psi)|^2}, \]

\[ \psi = \text{atan} \left( \frac{\zeta}{i} \right) \]
Ensemble averages

Frequency Dependance: \( \omega L/c \)

If \( L \) or \( c \) are (rectangle distributed)
Random Variables

1 element:

Ensemble Av. = Frequency Av.

\[ \omega (\eta + \eta_c) \langle E_1 \rangle_{k_oL} = \langle P_{in} \rangle_{k_oL} = \Pi_{in} \]
Exact Ensemble Averages
Two Elements

Random elements:

\[
(k_0 L)_i = \langle (k_0 L)_i \rangle + R[-\pi/2, \pi/2]
\]

\[
\langle P^{1,2}_{\text{coup}} \rangle_{(kL)_1,(kL)_2} = C_e \left( \langle \hat{e}_1 \rangle_{(kL)_1,(kL)_2} - \langle \hat{e}_2 \rangle_{(kL)_1,(kL)_2} \right)
\]

\[
C_e = \frac{C}{Q - C/M_1 - C/M_2}, \quad C = \frac{2 \operatorname{Re}(\xi)}{\pi 1 + |\xi|^2},
\]

\[
Q = \sqrt{1 + \frac{2\pi C}{\tanh(\pi M_1) \tanh(\pi M_2)} + \left( \frac{\pi C}{\tanh(\pi M_1)} \right)^2 + \left( \frac{\pi C}{\tanh(\pi M_2)} \right)^2 - (\pi C)^2}
\]

- Proved CPP for ensemble averages
- If \( \gamma < 1 \), the one-way approach is OK, otherwise the Connectivity depends on damping (and thus on coupling damping)
Conductivity for Exact Ensemble normalised with Travelling Wave Estimate

\[ M_2 = 10^*M_1 \]
Conductivity for Exact Ensemble normalised with Travelling Wave Estimate

Errors are not large if: $2C / (\pi M_1 M_2) < 1$

3 modes in a 1/3-octave band $\eta=0.01 \rightarrow M=0.12$
Comments on Coupling strength

• Dynamic coupling / Connection strength
  – Critical to get right, when using a one-way method, and, for coupling conditions, when using uncoupled modes

• Smith’s criterion $C \ll M_i$
  – Describes the character of SEA solutions
  – Does not validate an SEA model
  – If it is large, SEA might give the right answer for the wrong reason

• Modal interaction strength
  – Defines if response is given by local or global modes
  – If it is large, CLFs are smaller and might depend on damping
    • Experimentally verified for a ship structure (Nilsson 1978)
    • Critical as noise control measures are incorrectly predicted
  – Might validate Langley’s weak coupling criterion: Point mobility for uncoupled and coupled elements are equal
    • If so, we can measure point mobilities and check
  – Might be observed from the impulse response of a connected element