Statistical energy analysis made simple, and difficulties with strong coupling

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Statistical Energy Analysis

- “SEA: Seems Easy, Aint ”
  - (anon)

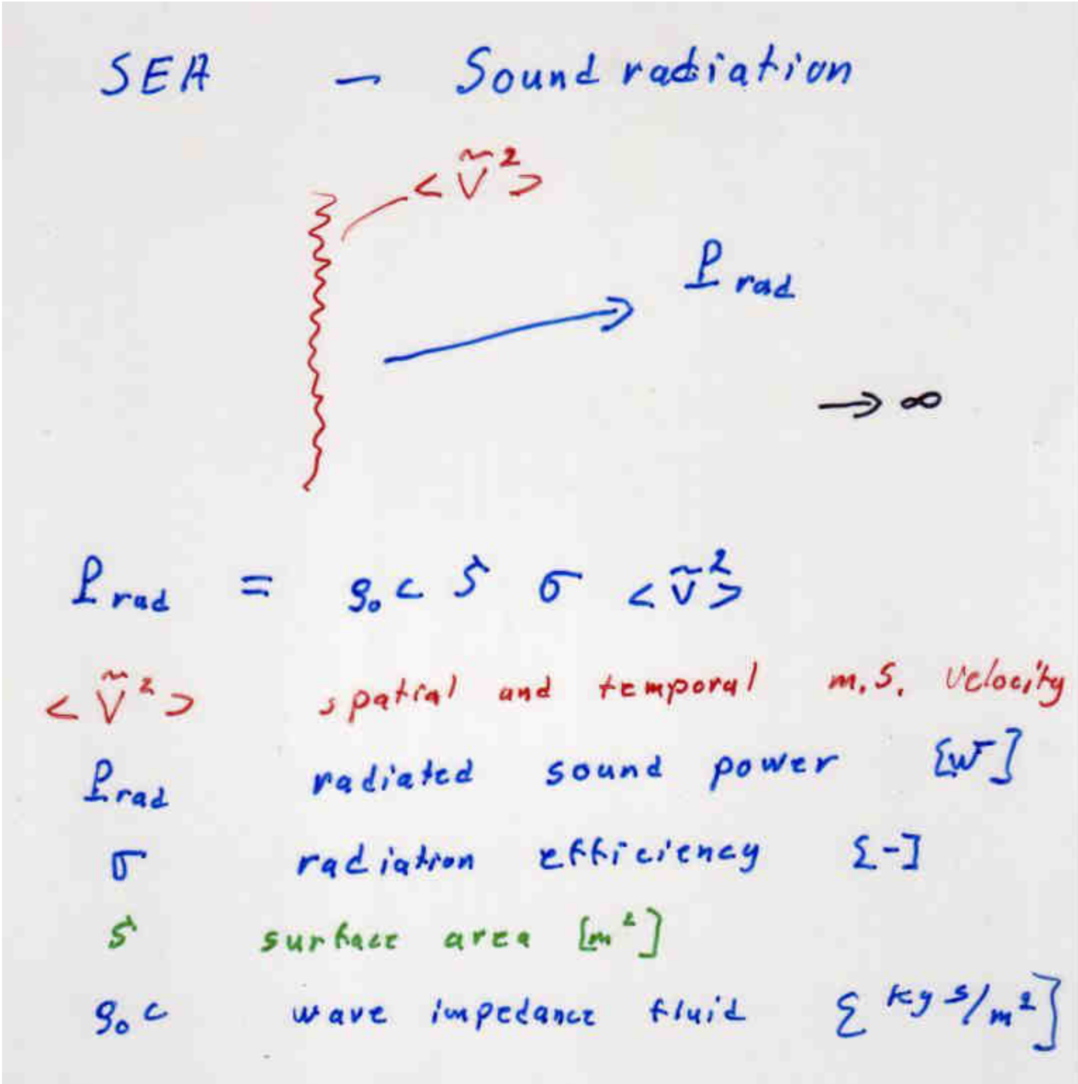
- “You must have faith” .. when using SEA
  - (Bob Craik)

- Don’t be intimidated
  - You have reasons to use SEA
  - SEA is difficult but no more so than, say, the FEM

Stockholm Concert Hall

- Play Cello on Floating Floor
  - Very trendy
  - Endpin induces floor vibrations
    - Radiated sound increase loudness
  - Risk Low Frequency Absorption
    - Too high already

https://open.spotify.com/track/3Mle6lLTYnFdNxb60USbgQ

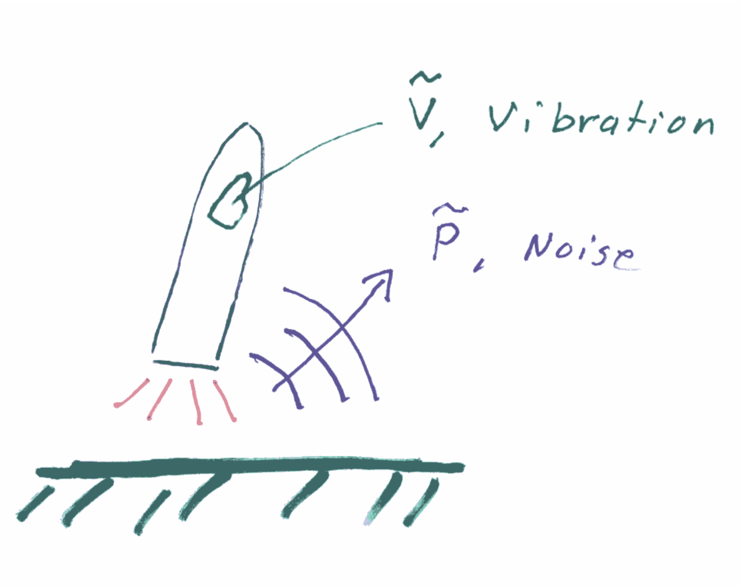


F. G. Leppington, E.G. Broadbent, K.H. Heron, The acoustic radiation efficiency of rectangular panels, Proc Roy Soc 382 (1982) 245-271.

$$\sigma = \sigma\left(k_a / k_s \, , k_a^2 S, \, k_a P\right)$$

k_a – Fluid wave number; k_s – Structure wave number

Acoustic Fatigue in Rocket-to-the-Moon



P. W. Smith 1962 *JASA* **34**, 640-647. Response and radiation of structural modes excited by sound:

$$M_s\,\hat{e}_s + C\left(\hat{e}_s - \hat{e}_a\right) = 0$$

$$C = \frac{\rho_o\, c\, n_s}{\mu_s}\sigma;\quad M_s = \left(\eta\,\omega\, n\right)_s;\quad n_s = \left(\frac{k\,S}{2\pi\, c_g}\right)_s$$

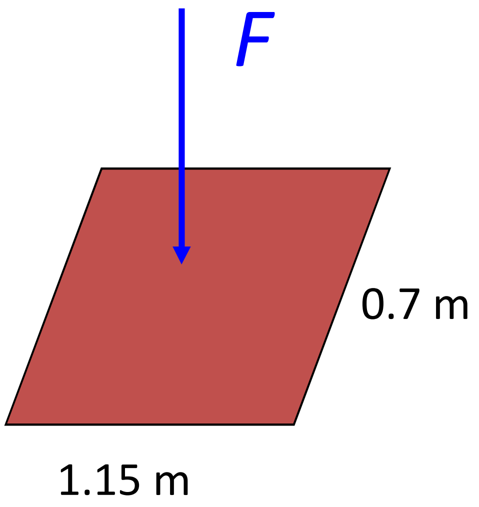
$$\hat{e}_i = \frac{E_i}{n_i};\quad \hat{e}_a = \left(\frac{2\pi^2\, c}{\rho_0\,\omega^2}\right)_a \left\langle \tilde{p}_a^2 \right\rangle;\quad \hat{e}_s = \left(\frac{2\pi\, c_g \mu}{k}\right)_s \left\langle \tilde{v}_s^2 \right\rangle$$

- The origin of SEA
- Some 200 000 modes in structure
- Acoustic field not quite known but it is power full

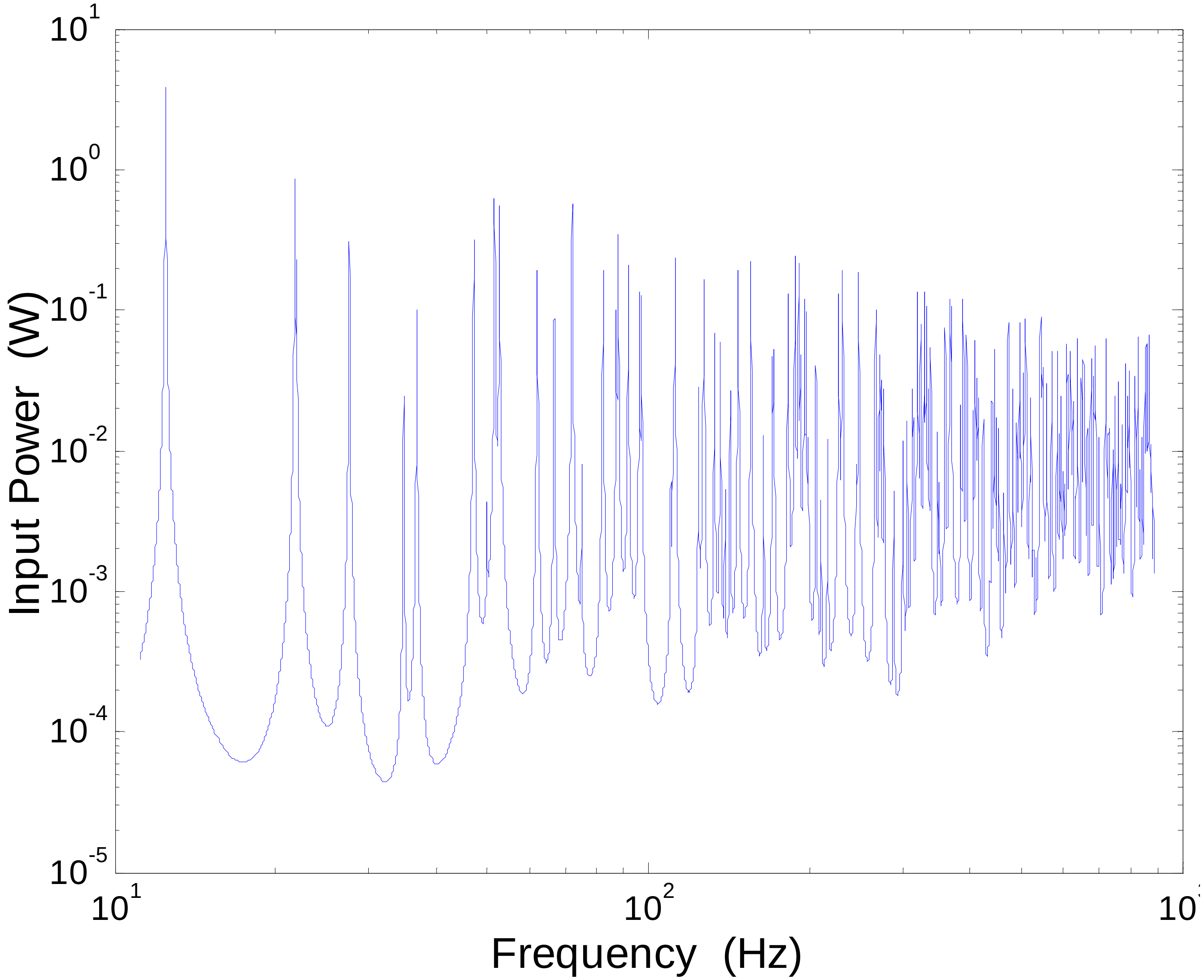
- "The principle of vibroacoustic reciprocity"
  - Relates sound radiation and sound reception

$$\eta_s\,?\;\; k_s\left(\omega\right)?$$

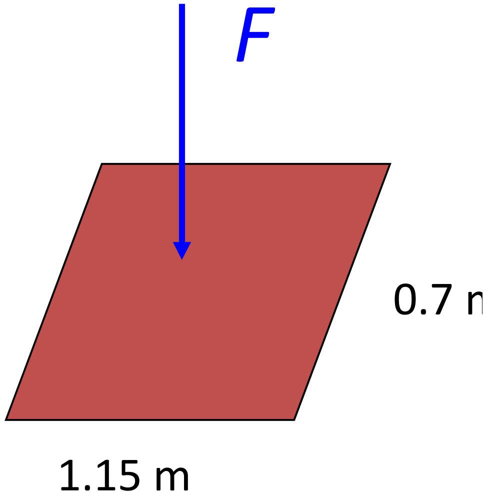
Simply Supported Plate



h = 1 mm



Simply Supported Plate ..



h = 1 mm + σ

σ = N(0, 20 μm)

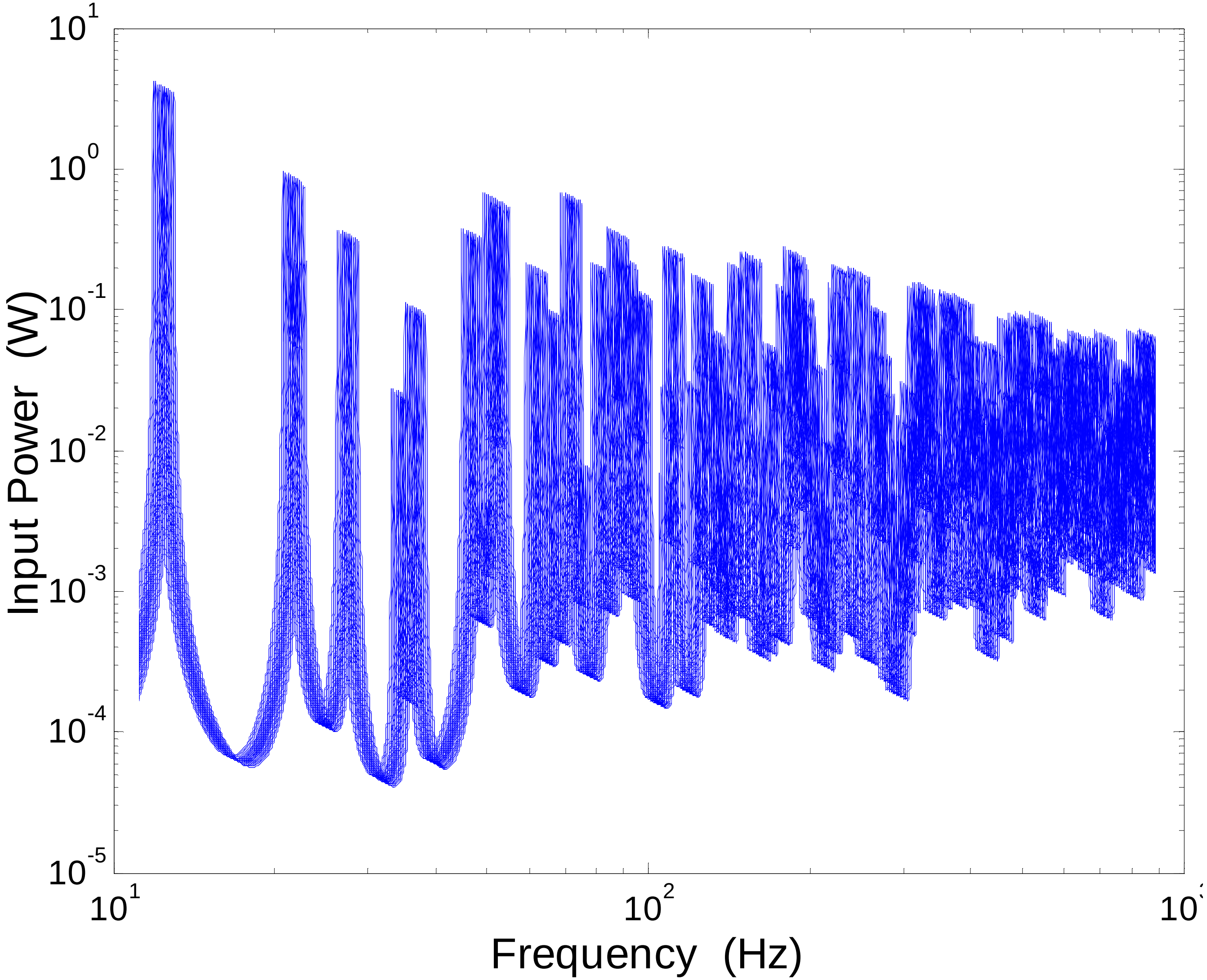
Acoustic limit:

$$\sigma_\omega \approx \delta\omega$$

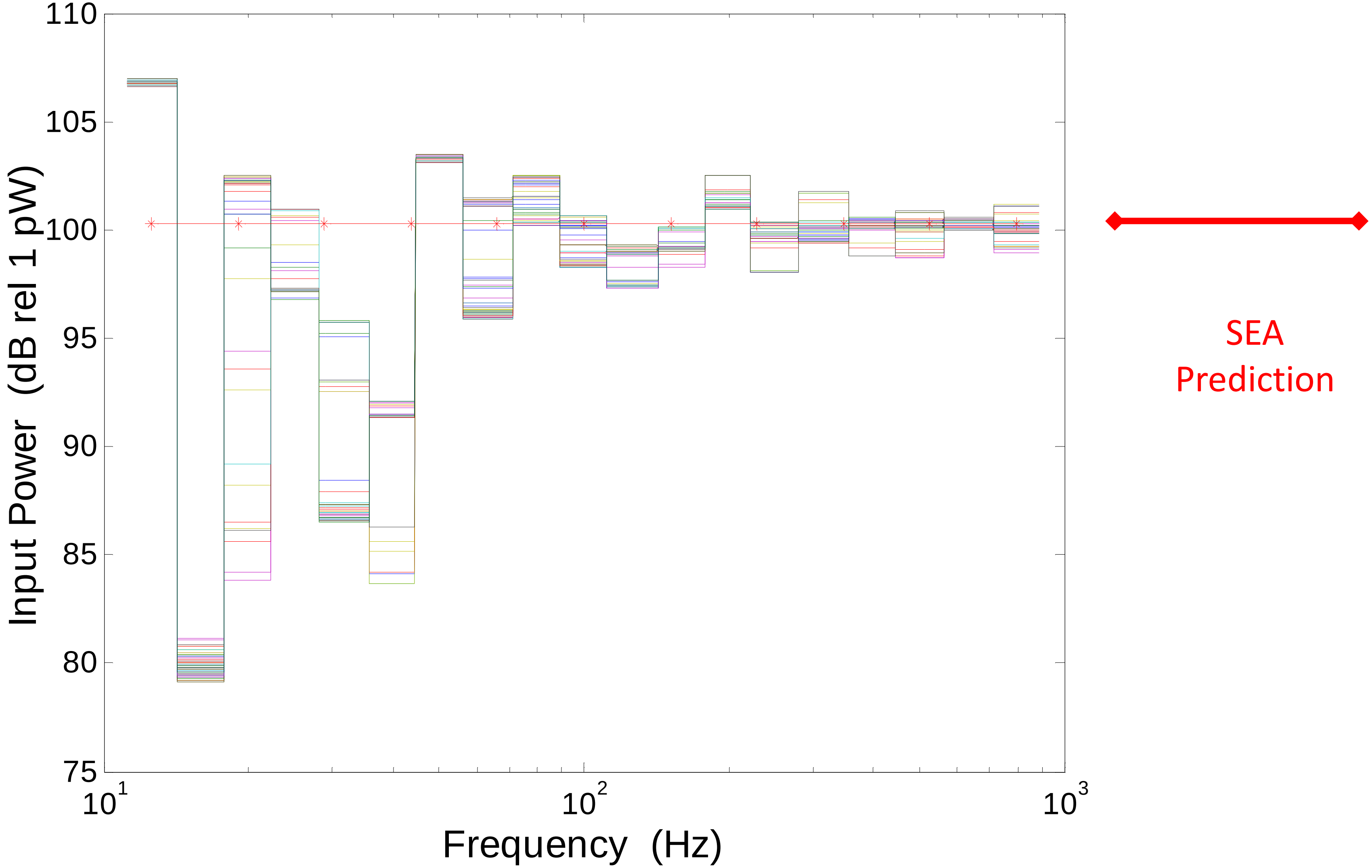
91Hz

100 realisations

$M \approx 1$ at 400 Hz



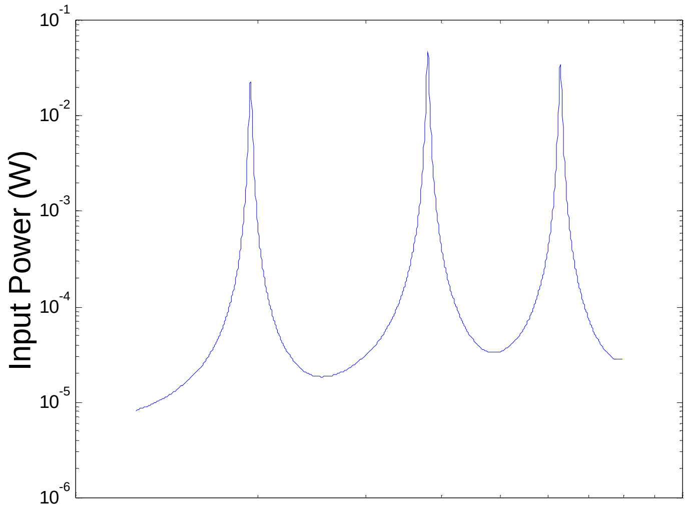
Simply Supported Plate ...

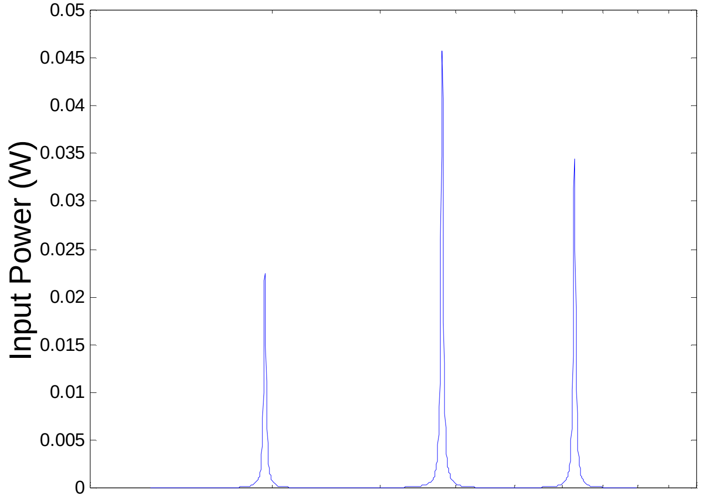


"Magic Integral"

$$\left\langle P_{in} \right\rangle_{x_o,\Delta\omega} = \frac{1}{\Delta\omega} \mathrm{Re} \int_{\Delta\omega} \sum_r \frac{-i\omega \left\langle \left| F\,\phi_r(x_0) \right|^2 \right\rangle_{x_0} d\omega}{m_r \left(\omega_r^2 - 2i\omega\omega_r\xi_r - \omega^2\right)}$$

$$\approx \left\langle \frac{\left|F_r\right|^2}{m_r} \right\rangle_r \frac{1}{\Delta\omega} \int_{\Delta\omega} \sum_r \frac{\pi}{2} \delta(\omega - \omega_r)\, d\omega \approx \frac{\pi}{2} \frac{\left|F\right|^2}{m} \frac{\Delta N}{\Delta\omega}$$





Magic Integral

$$I = \int_{\omega_l}^{\omega_u} \mathrm{Re}\left(\frac{i\omega}{\omega_r^2 - \omega^2 + i\omega\omega_r\eta}\right) d\omega$$

$$= \int_{\omega_l}^{\omega_u} \frac{\omega^2 \omega_r \eta}{(\omega_r^2-\omega^2)^2 + (\eta\omega\omega_r)^2} d\omega$$

Assume i) $\dfrac{\omega_r - \omega_l}{\eta\,\omega_r} >> 1$

$\dfrac{\omega_u - \omega_r}{\eta\,\omega_r} >> 1$ Most input power is in the frequency band

ii) $\omega_r^2 - \omega^2 = (\omega_r + \omega)(\omega_r - \omega)$

$\approx 2\omega_r(\omega_r - \omega)$

$\approx 2\omega(\omega_r - \omega)$

$\omega_r \approx \omega$, at frequencies for which the integrand is large

$$I \approx \frac{1}{4} \int_{\omega_l}^{\omega_u} \frac{\eta\,\omega_r}{(\omega_r - \omega)^2 + (\omega_r \eta / 2)^2}$$

$$= \frac{1}{2} \left[\operatorname{atan}\left(\frac{\omega - \omega_r}{\eta\,\omega_r} \right) \right]_{\omega_l}^{\omega_u}$$

$$\approx \frac{1}{2} \left[\operatorname{atan}(+\infty) - \operatorname{atan}(-\infty) \right]$$

$$= \pi / 2 \qquad \Rightarrow \langle P_{in} \rangle_{x_o, \Delta\omega} \approx \frac{k_s\, |F|^2}{4\, \mu_s\, c_g}$$

Input Power independent of: *i)* size, and *ii)* damping

Very Large Homogenous structure, excited at a Random Location.

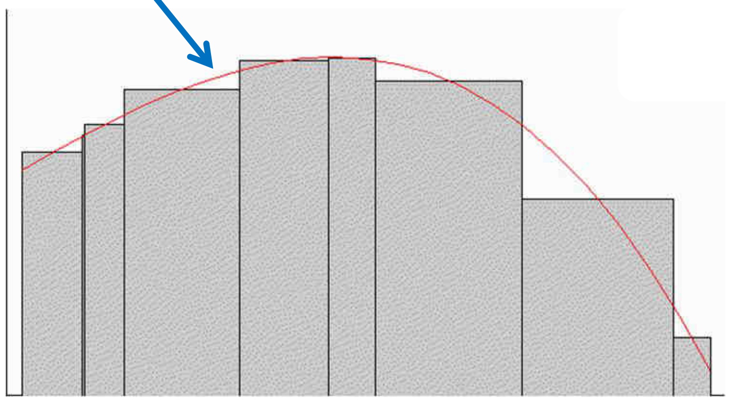
$$P_{in} = \mathrm{Re}\left(\sum_n \frac{-i\,\omega \left|\mathbf{F_n}\right|^2}{m_n\left(\tilde{\omega}_n^2 - \tilde{\omega}^2\right)} \right)$$

$$\approx \frac{\left|\mathbf{F}_0\right|^2}{m} \mathrm{Re}\left(\sum_n \frac{-i\,\omega}{\left(\tilde{\omega}_n^2 - \tilde{\omega}^2\right)} \right) = \frac{\left|\mathbf{F}_0\right|^2}{m} \frac{}{\delta\omega_n} \mathrm{Re}\left(\sum_n \frac{-i\,\omega}{\left(\tilde{\omega}_n^2 - \tilde{\omega}^2\right)} \delta\omega_n \right)$$

$$\approx \frac{\left|\mathbf{F}_0\right|^2}{m} \frac{1}{\delta\omega} \mathrm{Re}\left(\int \frac{-i\,\omega}{\left(\tilde{\omega}_n^2 - \tilde{\omega}^2\right)} \,\mathrm{d}\,\omega_{\mathrm{n}} \right)$$

$$= \frac{\pi}{2} \frac{n}{m} \left|\mathbf{F}_0\right|^2$$

Riemann Sum,



Valid approximation if M > 1

Random structure

Random eigenfrequencies. Probability density: $p(\omega_n)$

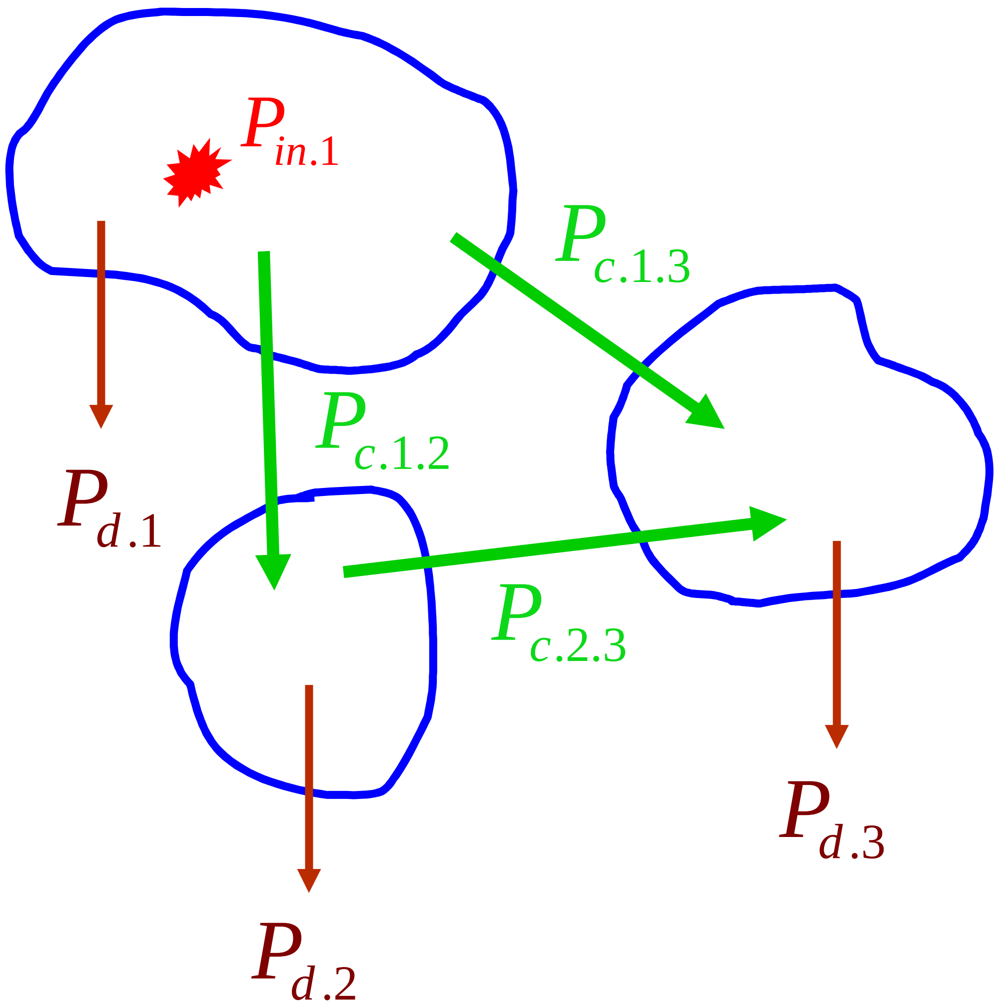
$$P_{in} = \int \sum_n p(\omega_n)\,\mathrm{Re}\left(\frac{-i\,\omega\left|\mathbf{F_n}\right|^2}{m_n\left(\tilde{\omega}_n^2 - \tilde{\omega}^2\right)}\right)\mathrm{d}\,\omega_n$$

Rectangle distributed: $p(\omega_n) = \begin{cases} \dfrac{1}{\omega_u - \omega_l}, & \omega_l < \omega_n < \omega_u \\ 0 & \text{otherwise} \end{cases}$

$$P_{in} = \sum_n \frac{1}{\omega_u - \omega_l}\int_{\omega_l}^{\omega_u} \mathrm{Re}\left(\frac{-i\,\omega\left|\mathbf{F_n}\right|^2}{m_n\left(\tilde{\omega}_n^2 - \tilde{\omega}^2\right)}\right)\mathrm{d}\,\omega_n \approx \frac{\pi}{2}\left\langle \frac{\left|\mathbf{F_n}\right|^2}{m_n} \right\rangle_n \frac{\Delta N}{\Delta\omega}$$

One substructure: ergodicity

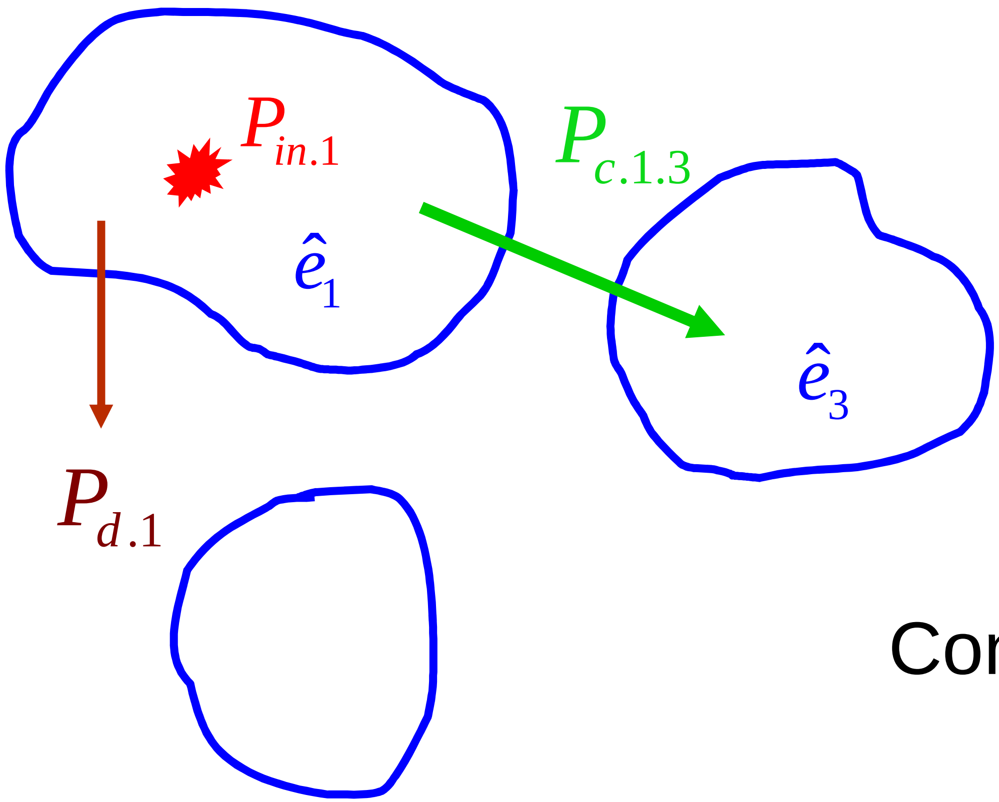
SEA Formulation



Power Balance:

$$P_{in.i} = P_{d.i} + \sum_{j \neq i} P_{c.i.j}$$

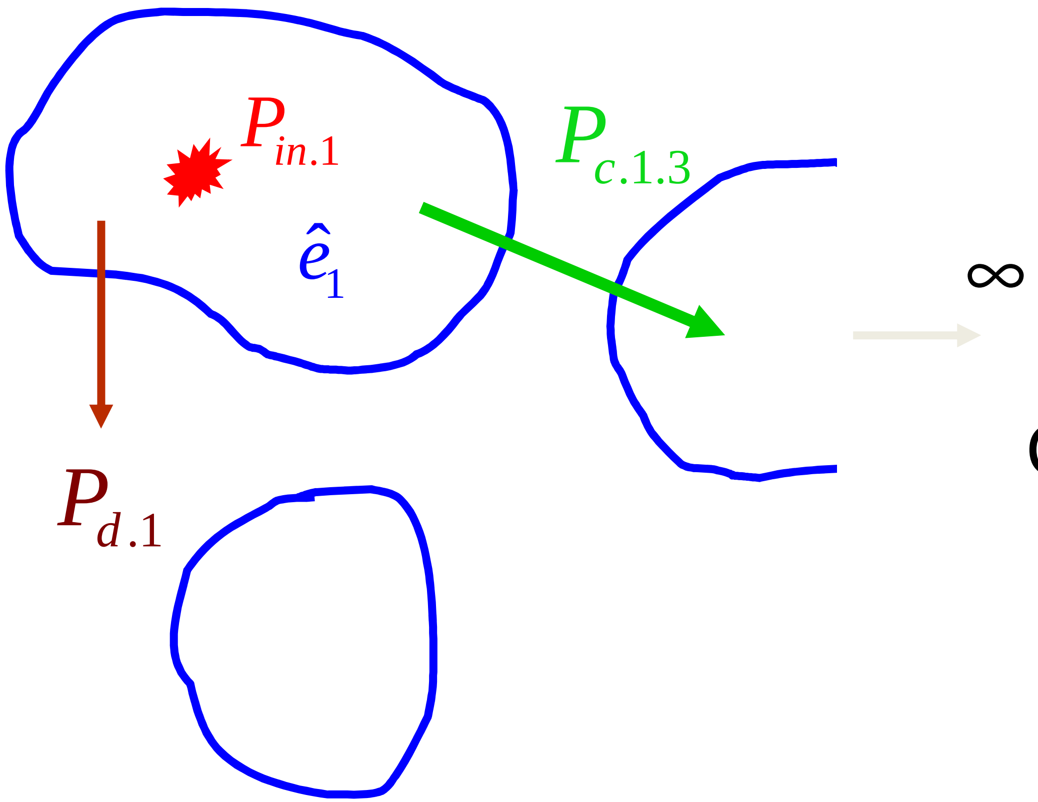
"Compact Support"



Conductivity given by:

$$P_{c.1.3} = C^{1.3}\left(\hat{e}_1 - \hat{e}_3\right)$$

One – Way Estimate



Conductivity given by:

$$C^{1.3} = \left(\frac{P_{c.1.3}}{\hat{e}_1} \right)$$

i.e., we use this Eq. to calculate the conductivity

SEA Formulation ...

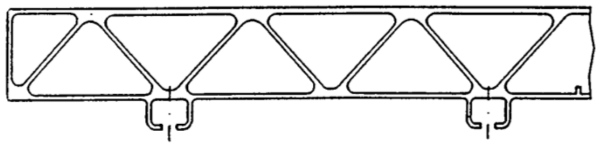
$P_{d.i} = \eta_i \, \omega E_i = M_i \, \hat{e}_i$	Linear proportional damping
P_{in}	Independent of Connected Elements
$P_{coup}^{i,j} = C^{i,j} \left(\hat{e}_i - \hat{e}_j \right)$	Coupling Power Proportionality (CPP)
$\hat{e}_i = E_i / n_i$	Modal Power
n	Modal Density
$M = \eta \, \omega n$	Modal Overlap Factor
$C^{i,j} = \eta_{coup}^{i,j} \, \omega n_i$	Conductivity

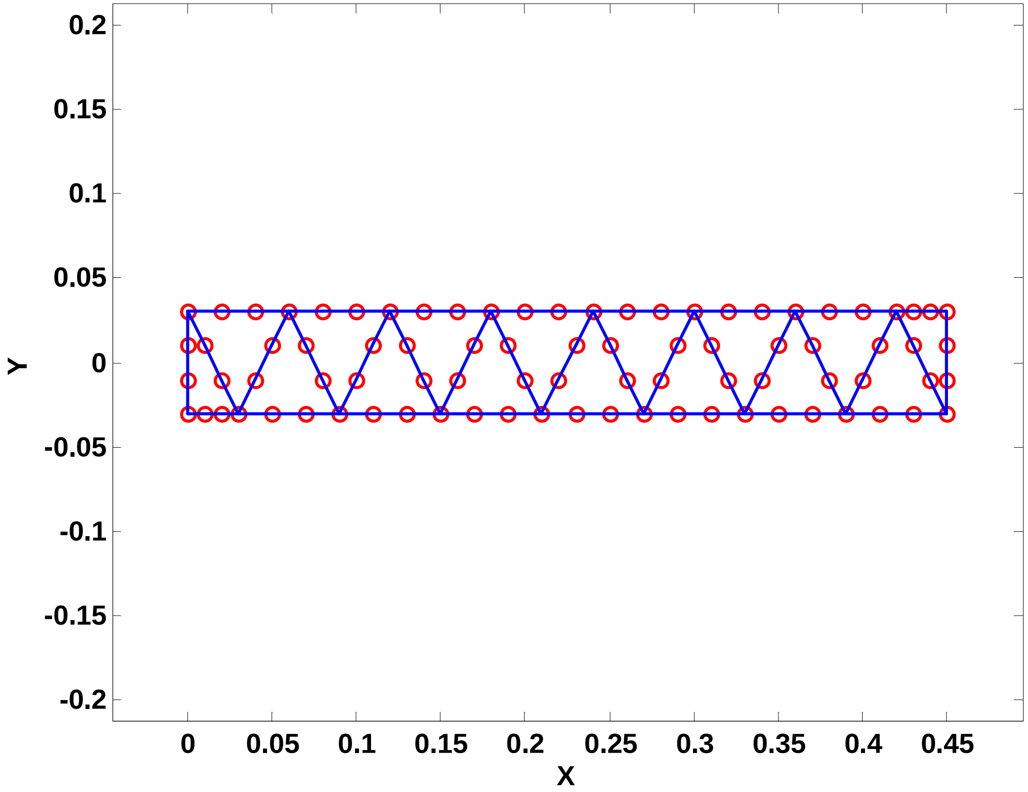
Yuet-Yan Pang
"Air-Borne Sound Transmission through Extruded Profiles"

- **Air-borne sound transmission from bogie to interior – medium and high frequencies.**

Intercity train

2-D FE – Model





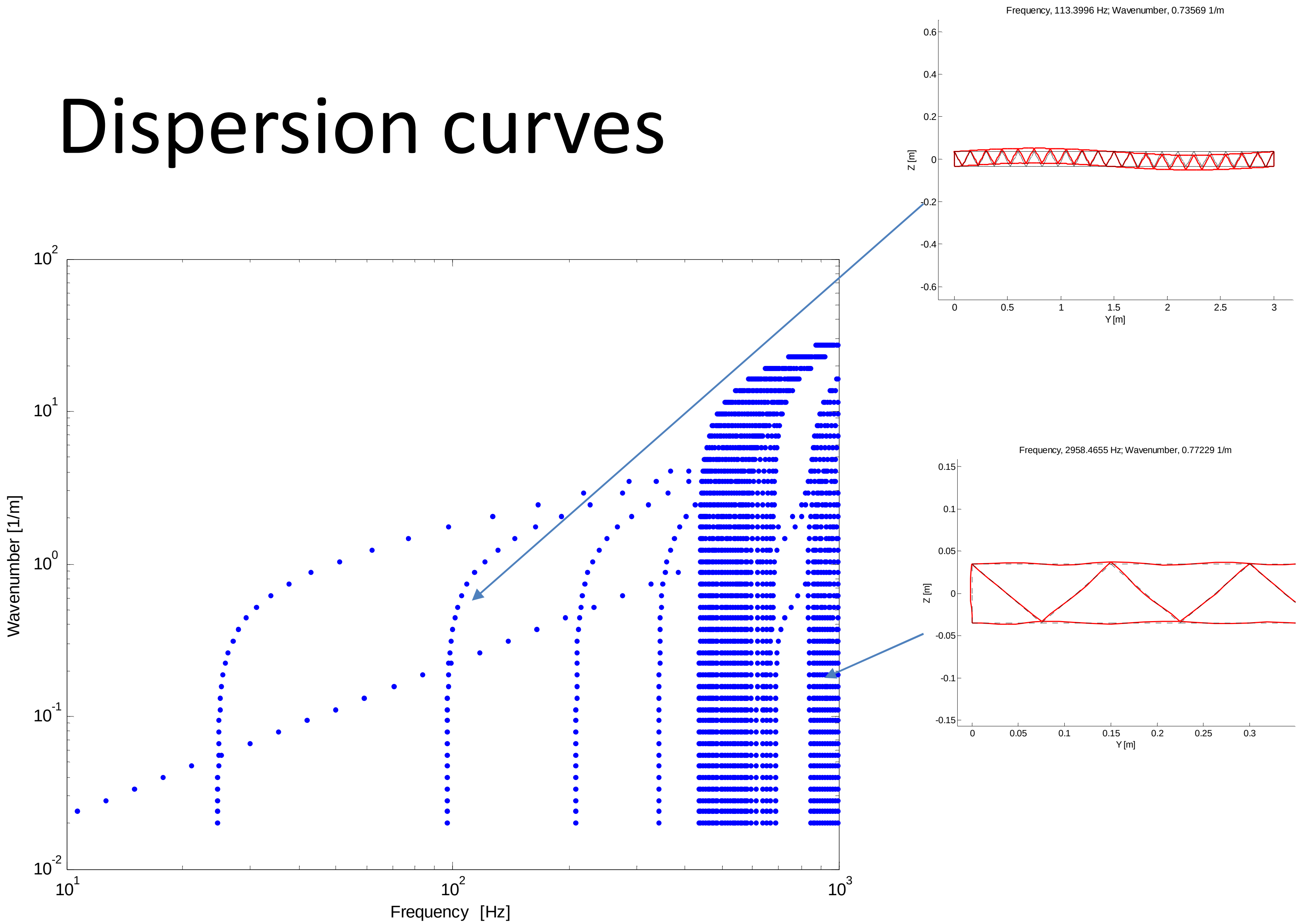
$$U(x,y,z,t)=$$

$$=\mathrm{Re}\left(\left[\boldsymbol{\Psi}(x,y)\right]^{\mathrm{T}}\mathbf{V}(z)\;\mathrm{e}^{-\mathrm{i}\omega t}\right)$$

$\boldsymbol{\Psi}$ – FE - Shape Functions

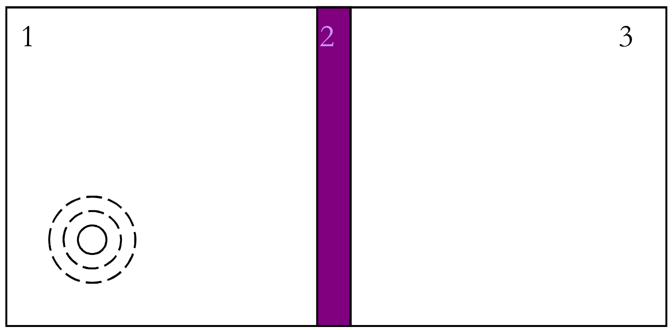
$\mathbf{V}$ – Nodal Displacements

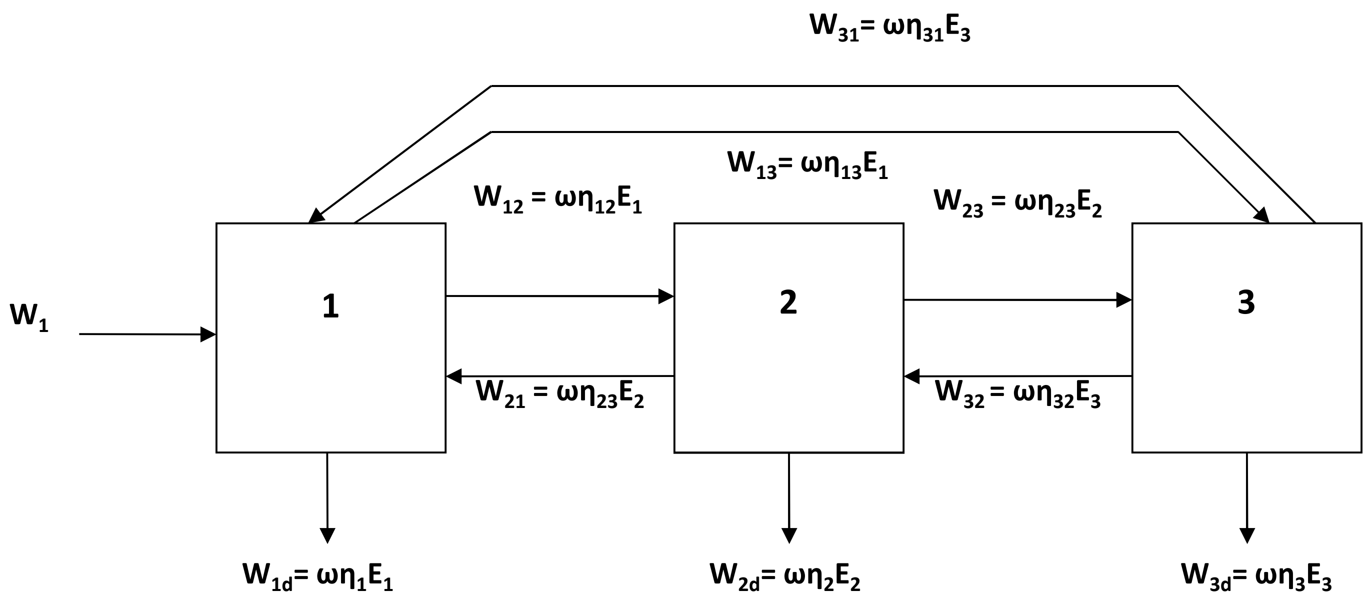
Dispersion curves



Statistical Energy Analysis

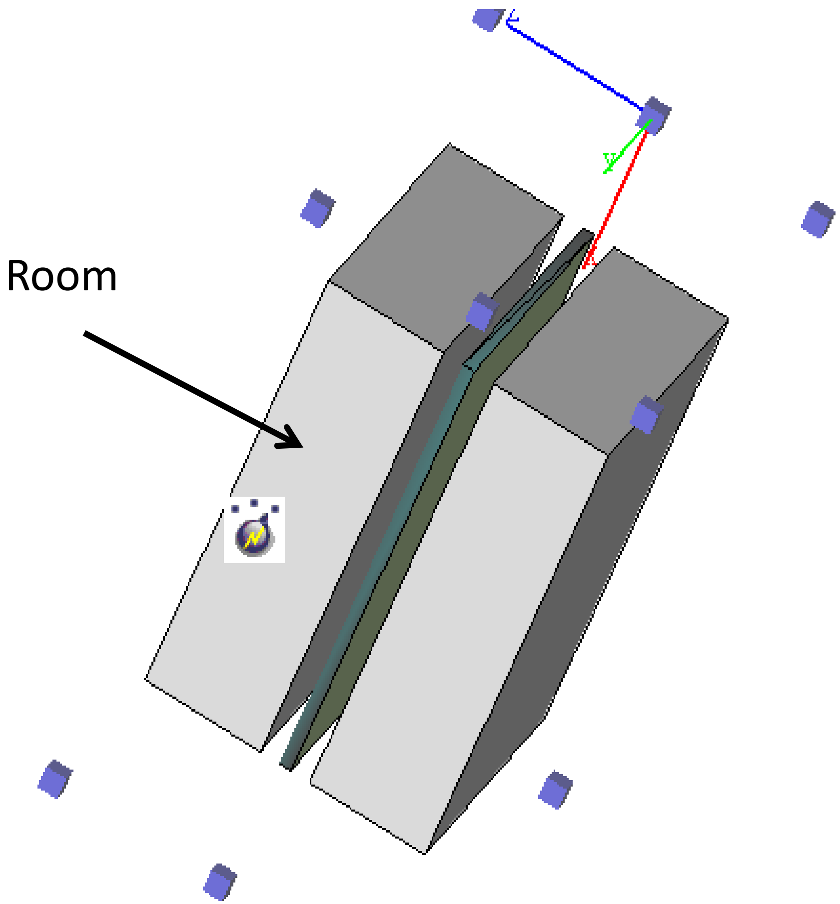
Low Frequency Model

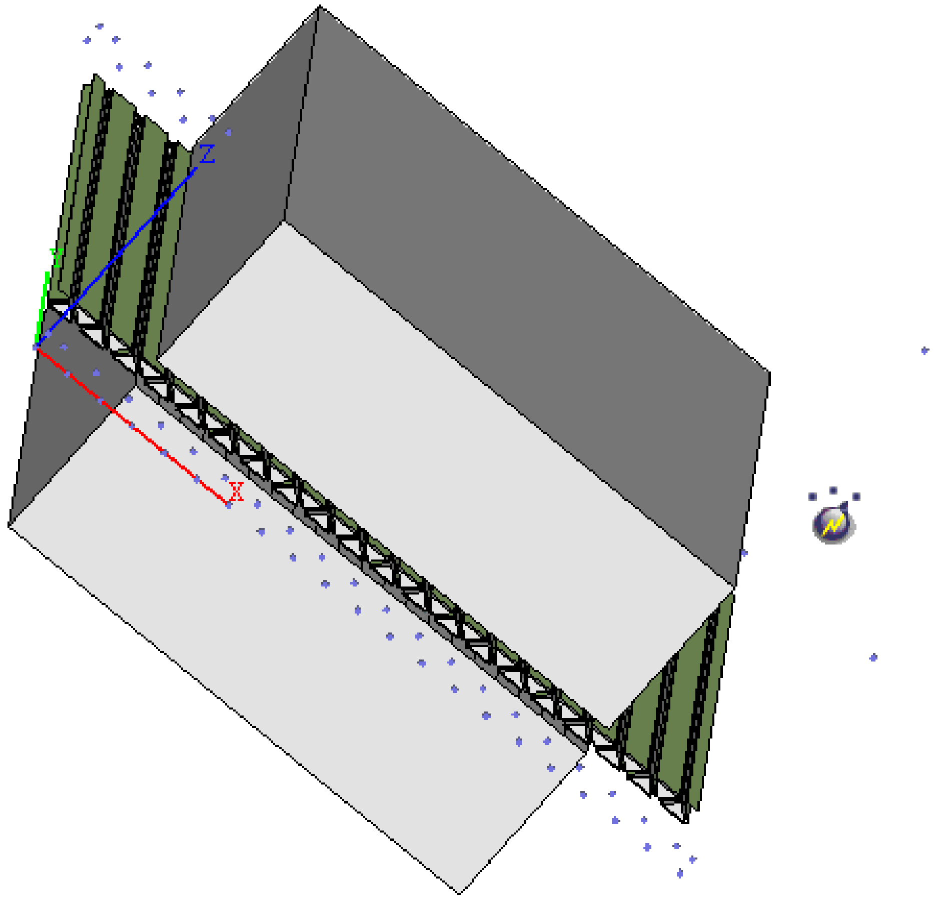




Two AutoSEA Models

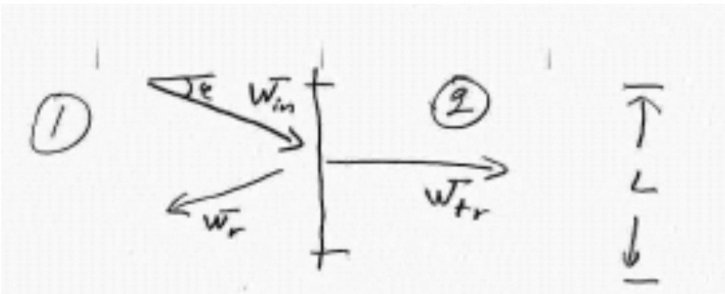
An orthotropic plate – Many Plates

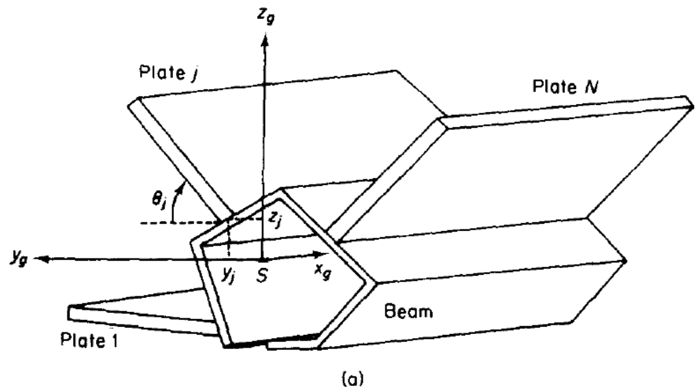




Plates line-coupled along a beam

RS Langley, KH Heron *JSV* 1990
"Elastic wave transmission
through beam/plate junctions"



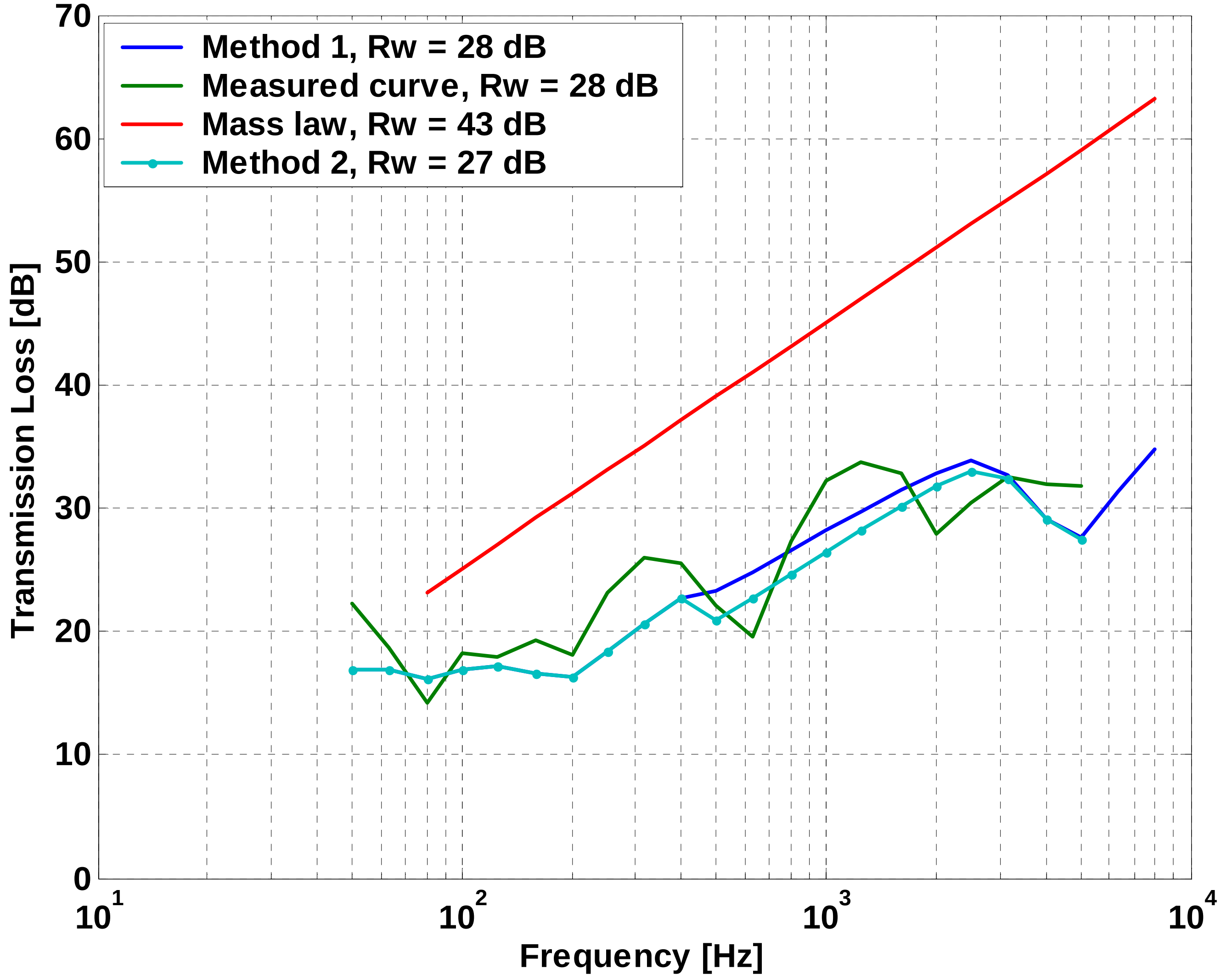


Diffuse Field in Element 1:

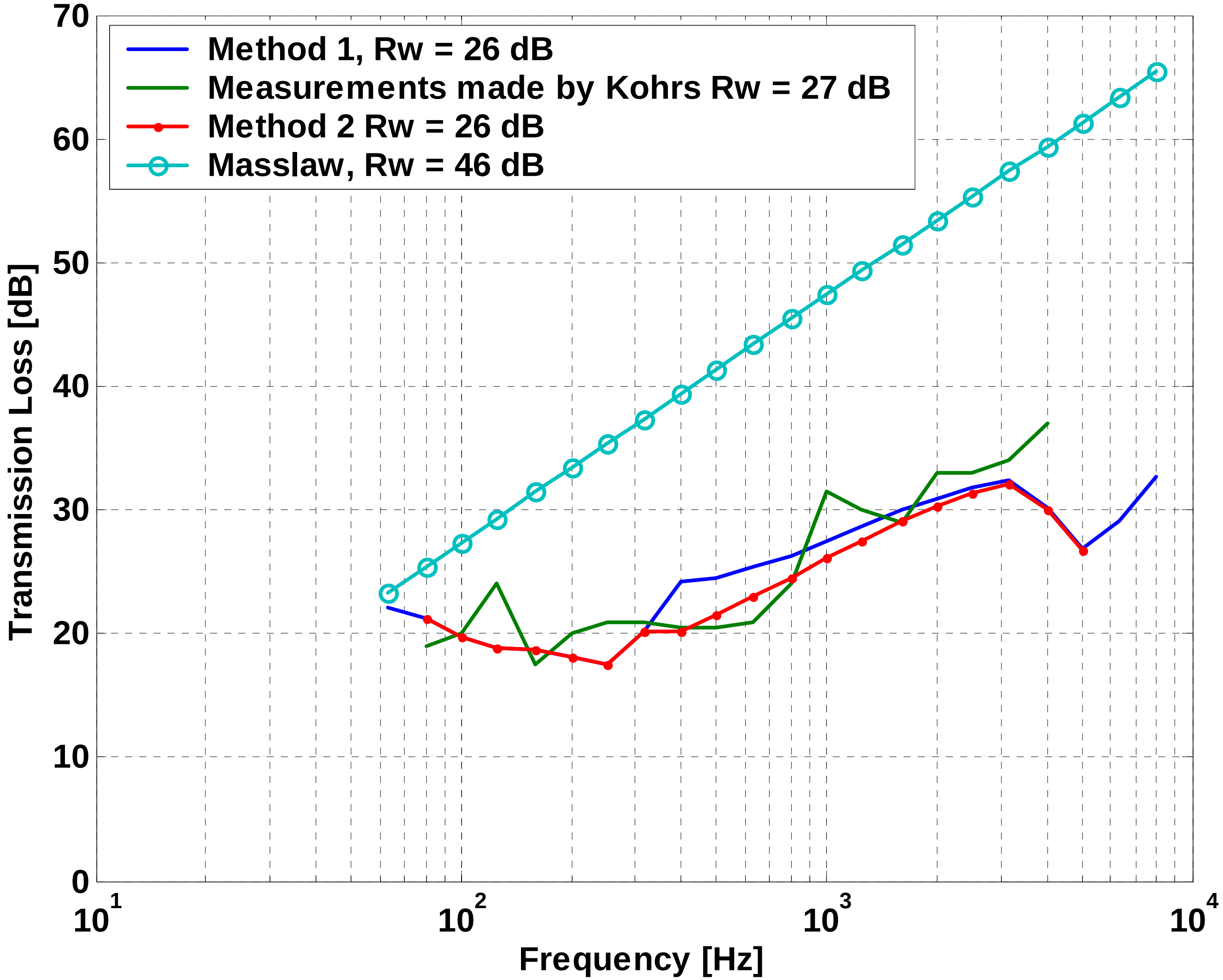
$$\tau(\phi) = \frac{W_{tr}(\phi)}{W_{in}(\phi)}$$

$$C = \frac{k_1 L}{4\pi} \left\langle \tau(\phi) \right\rangle_{\phi}$$

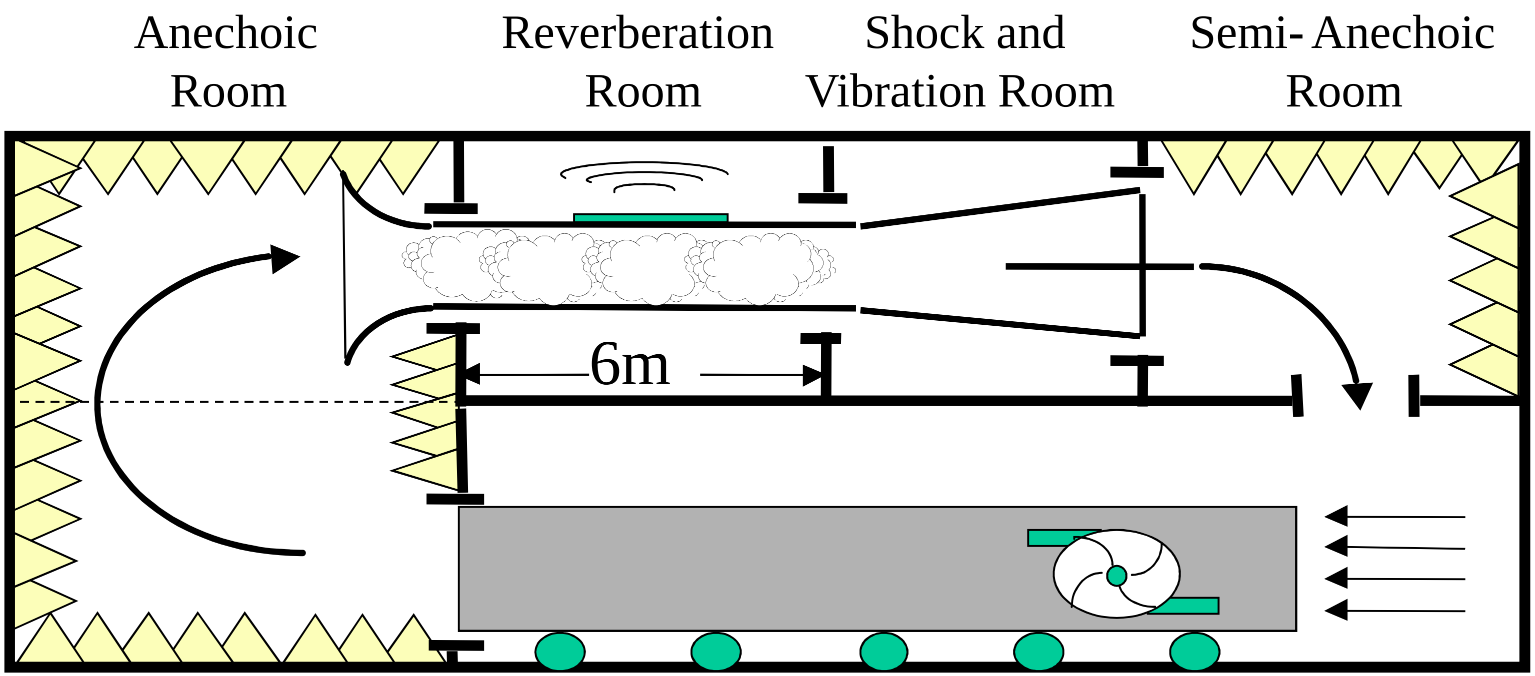
Transmission Loss Result for Metro train floor



Transmission Loss Result for Regional train



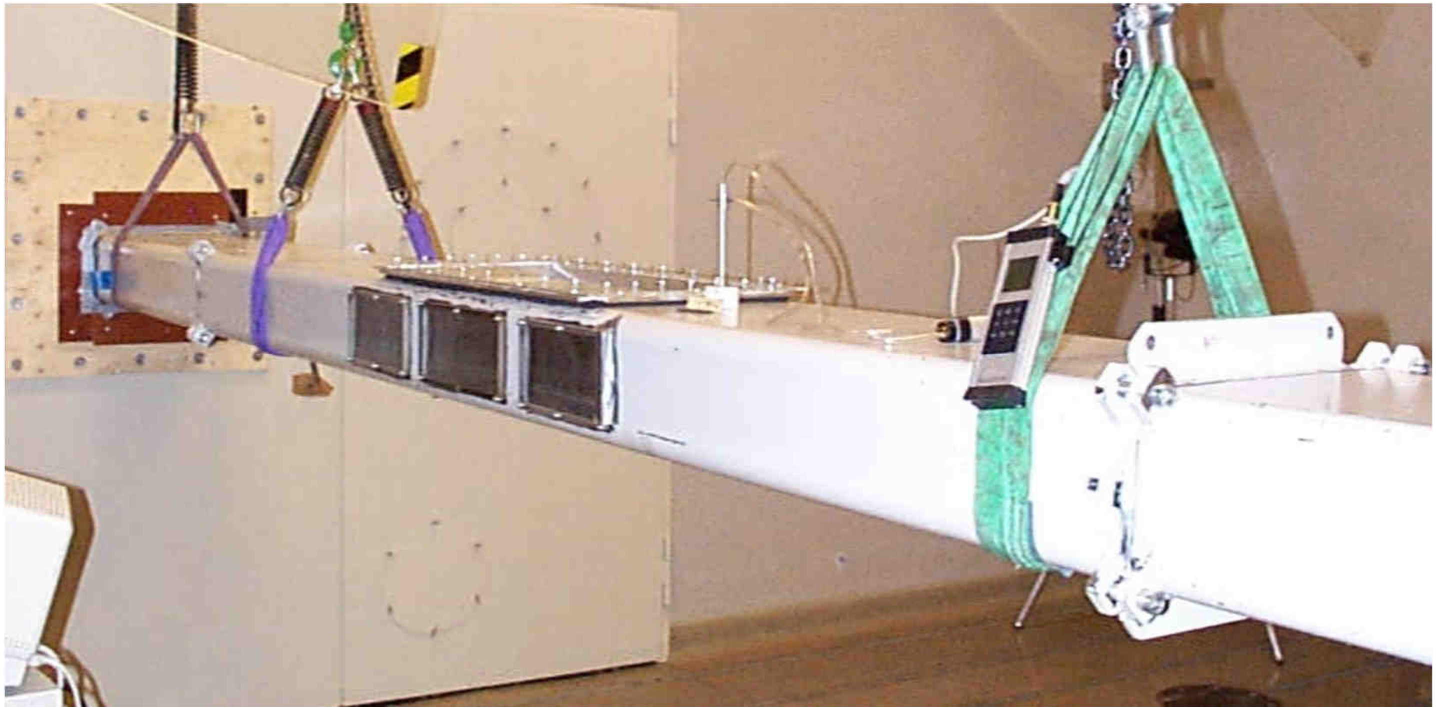
KTH-Enable Experiments



V_{max} = 130 m/s

Background Noise < 25 dB

Measurement Setup



- Plate vibration caused by
  – Turbulence?
  – Tunnel vibration?

Vibrations of Plate and Tunnel

Sufficient difference?

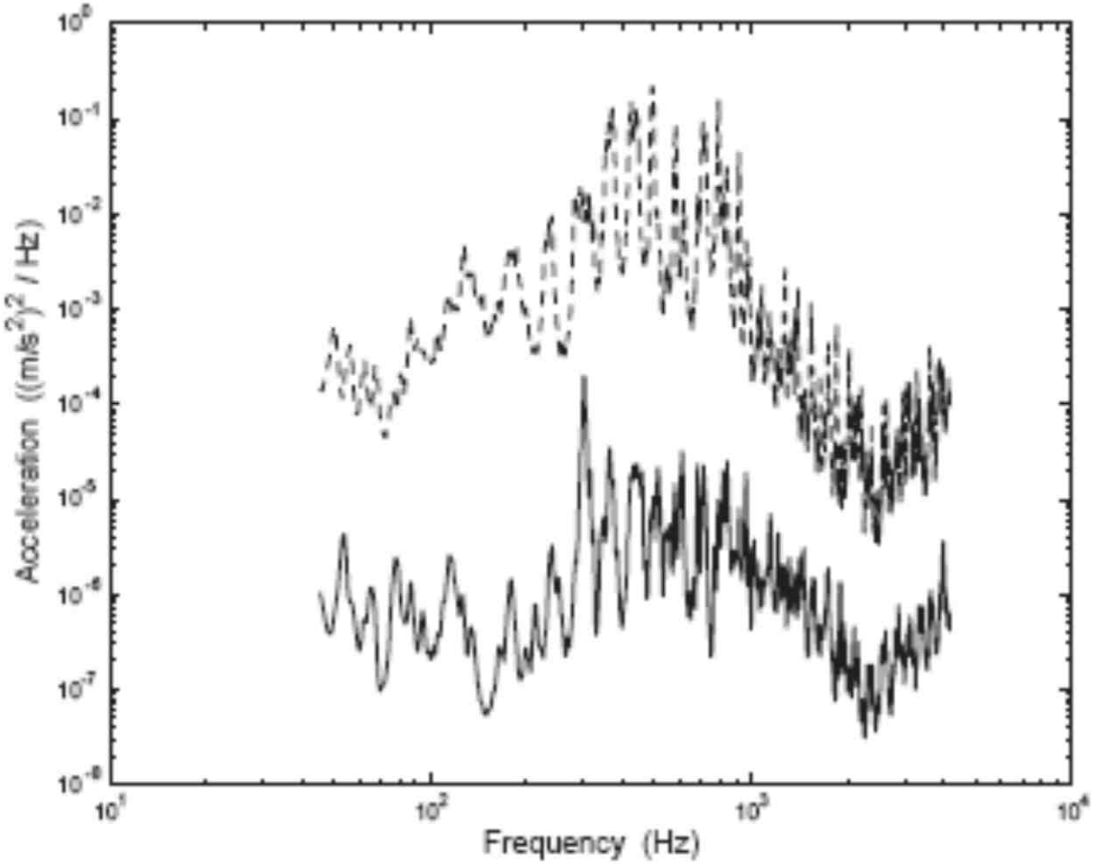
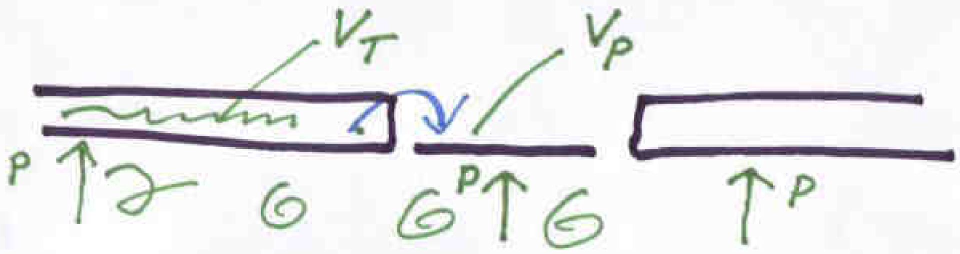


Fig. 15. Accelerations with 120 m/s flow speed in original wind tunnel. —, tunnel; - - -, test plate.

Example: Wind tunnel



V_p caused by p or V_T ?

1, Tunnel only is excited, find

$$H_T \equiv \left(\frac{\langle V_p^2 \rangle}{\langle V_T^2 \rangle} \right)_T$$

2, Under operation, measure

$$H_M \equiv \left(\frac{\langle V_p^2 \rangle}{\langle V_T^2 \rangle} \right)_M$$

3, If $H_T \ll H_M$ OK $\quad H_T$?

SEA, tunnel excited

$$\begin{bmatrix} M_T + \tau & -\tau \\ -\tau & M_p + \tau \end{bmatrix} \begin{bmatrix} \hat{e}_T \\ \hat{e}_p \end{bmatrix} = \begin{bmatrix} P_{in} \\ 0 \end{bmatrix}$$

Whatever the values of M_T, M_p and τ:

$$\hat{e}_p < \hat{e}_T$$

$$m_p \langle \tilde{v}_p^2 \rangle / n_p < m_T \langle \tilde{v}_T^2 \rangle / n_T$$

$$m_p = \rho_p t_p S_p \quad , \quad m_T = \rho_T t_T S_T$$

$\therefore$ Conservativ Estimate of H_T

$$H_T \equiv \frac{\langle \tilde{v}_p^2 \rangle_T}{\langle \tilde{v}_T^2 \rangle_T} < \frac{m_T}{m_p} \cdot \frac{n_p}{n_T}$$

If $H_M >> \frac{m_T}{m_p}\frac{n_p}{n_T} > H_T$ then OK

Need modal densities

Thin - walled plate :

$$\frac{m_p}{n_p} = \frac{\rho_p S_p t_p}{S_p / (3{,}6\, c_L t_p)} = 3{,}6 \left(\rho c_L t_p^2 \right)_p$$

=

Tunnel: 12 mm steel ; Plate: 1,6 mm Aluminium

=

At lower frequncies, tunnel as a beam

=

At higher frequencies, tunnel as a plate assembly

$$\frac{m_T}{n_T} = 3{,}6 \left(\rho c_L t^2 \right)_T$$

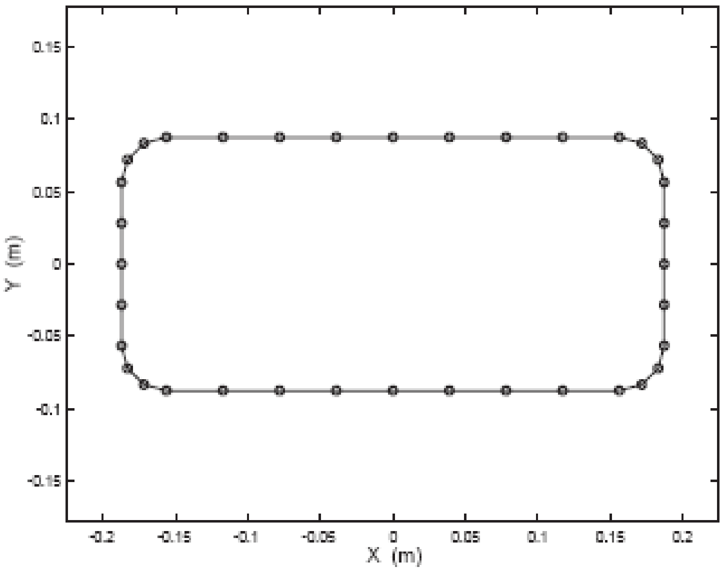
$$10 \log (H_T) < 10 \log \left[\frac{(\rho t^2)_T}{(\rho t^2)_p} \right] = \underline{22 \text{ dB}}$$

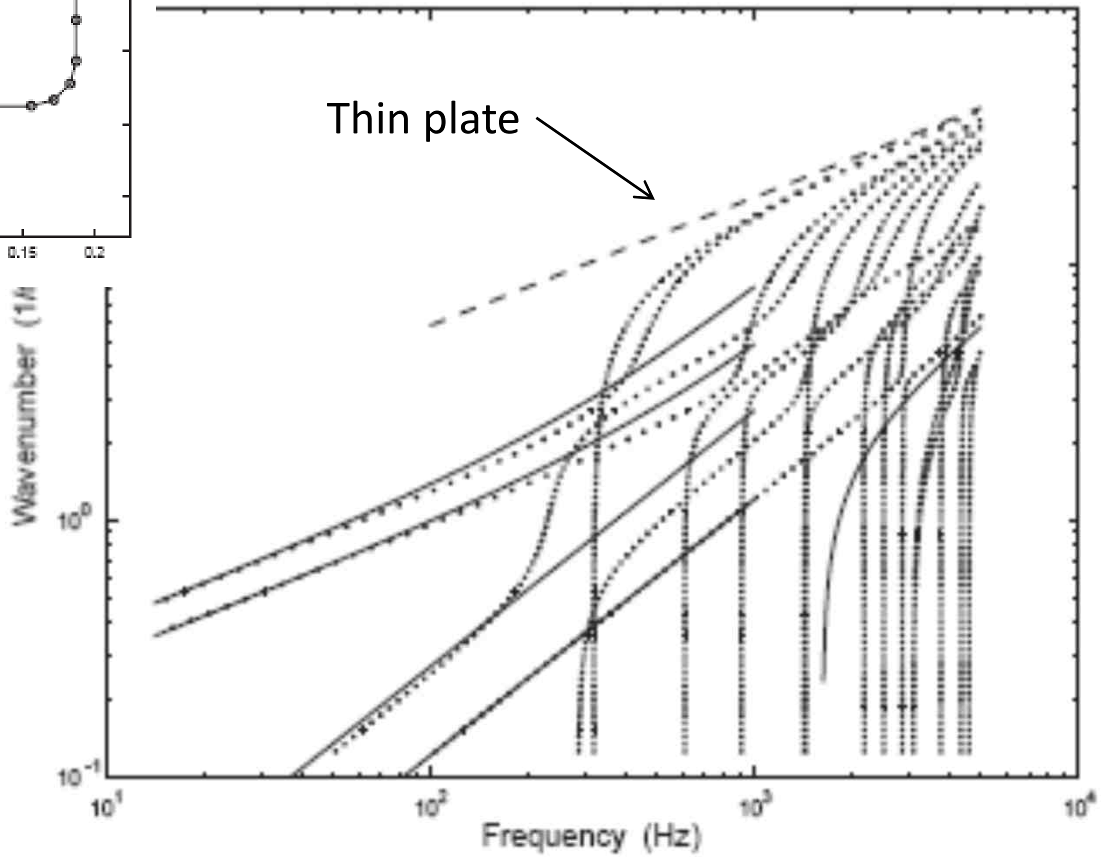
=

High / Low Frequencies ?

Other Frequencies ?

Waves in Tunnel





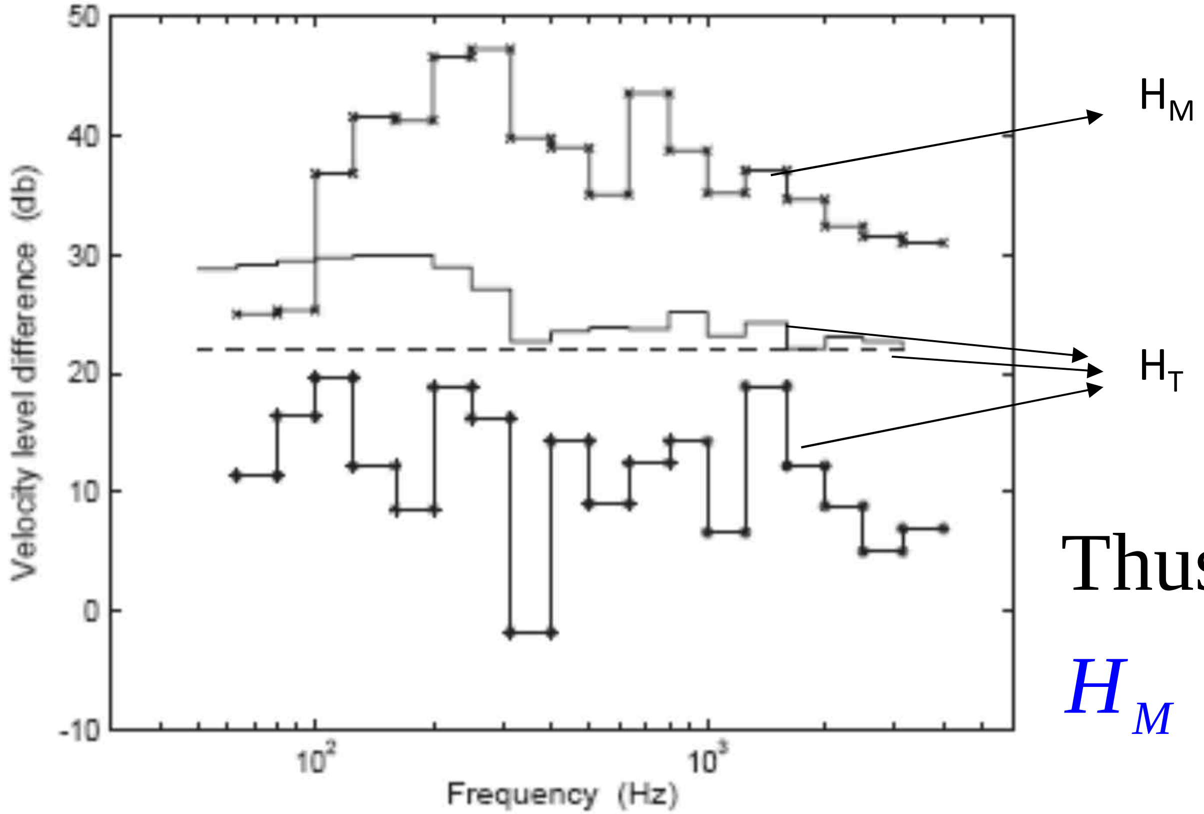
For each wave, r:

$$n_r = \frac{L}{\pi c_{g,r}} = \frac{L}{\pi}\frac{\partial k_r}{\partial \omega}$$

$$\left(\frac{n}{m}\right)_T = \frac{1}{\pi (\rho S)_T} \sum_r \frac{\Delta k_r}{\Delta \omega}$$

Final Result

- Viscoelastic damping on tunnel (improves at high frequencies)
- Blocking masses (solves 300 Hz problem)



Thus:

$$H_M >> H_T$$

Fig. 16. Velocity level difference between test plate and wind tunnel for modified wind tunnel. — × —, measured with 120 m/s flow; ——, conservative SEA estimate for when tunnel only is excited; - - - -, 22 dB; — * —, measured when tunnel is excited with a shaker.

Conclusions

- SEA elements are elements of response NOT elements of substructures
- SEA is built upon assertions of the vibroacoustic field in classes of structures:
  - Diffuse field in a room
  - Reverberant motion of plate with uncertain properties
  - ..
- SEA software are libraries for such “templates”
  - Trick is to know when and how to use them
    - Diagnostic measurements
    - Alternative calculations
    - Experience

- Wind tunnel: Just saying “SEA works” -> conservative upper bound
- Railway car: Once the templates are identified, the rest is easy
- Concert hall: Given η and k for floor, it’s easy to estimate
  - Sound radiation
  - Low frequency sound absorption

- SEA is very useful
- “Standard” SEA is built upon One-Way procedures for CLFFs

One-Way procedures for CLFs

- One junction at a time (compact support)
- Define field in first element
  - Level $\hat{e}$, “diffuse wave field”, “resonant modes”, …
- Express P_{tr}
  - Possibly, for “Weak Coupling”
  - Possibly, for infinite receiving element
  - Possibly, for element with random properties
- Express coupling loss

$$C_{1,2} = \omega n_1 \eta_c^{(1,2)} = P_{tr}^{(1,2)} / \hat{e}_1$$

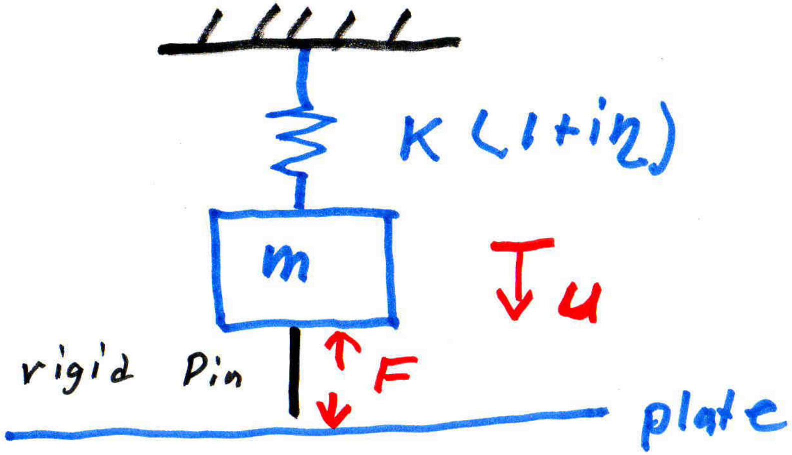
Examples of One-Way procedures for CLFs

- ISO standards for Sound Reduction Index ensures the conditions for One-Way procedures
- Sound Reception (Smith 1962)
- Sound radiation (Maidanik 1962, Leppington 1982)
- Walls and Double walls (Price 1970, Craik 2003, Finnveden 2007)
- Point coupling (Lyon 1975)
- Structural line coupling (Gibbs 1974, Langley 1990)
- …
- Shorter & Langley (2005) General Smith theory
- Le Bot (2007) Radiative exchanges

All of these: "Vibroacoustic Reciprocity"

$$C_{1,2} = C_{2,1}$$

Example: Cello exciting floor



$$\left(k\left(1+\mathrm{i}\eta\right)-m\omega^2\right)\tilde{u}=-\tilde{F}$$

$$F=Z_{plate}\,\mathrm{i}\,\omega\tilde{u}$$

$$m\left(\omega_o^2-\omega^2-\omega\,\mathrm{Im}\left(Z_{plate}/m\right)\right)\tilde{u}$$

$$+\mathrm{i}\,\omega_o^2 m\left(\eta+\eta_c\right)\tilde{u}=0$$

$$\eta_c=\omega\mathrm{Re}\left(Z_p\right)/\omega_o^2\,m$$

One mode: $n=\dfrac{1}{\Delta\omega}$; $\Delta\omega$ – Analysis band width

Conductivity: $C=\omega n\eta_c\approx\mathrm{Re}\left(Z_p\right)/\Delta\omega m$

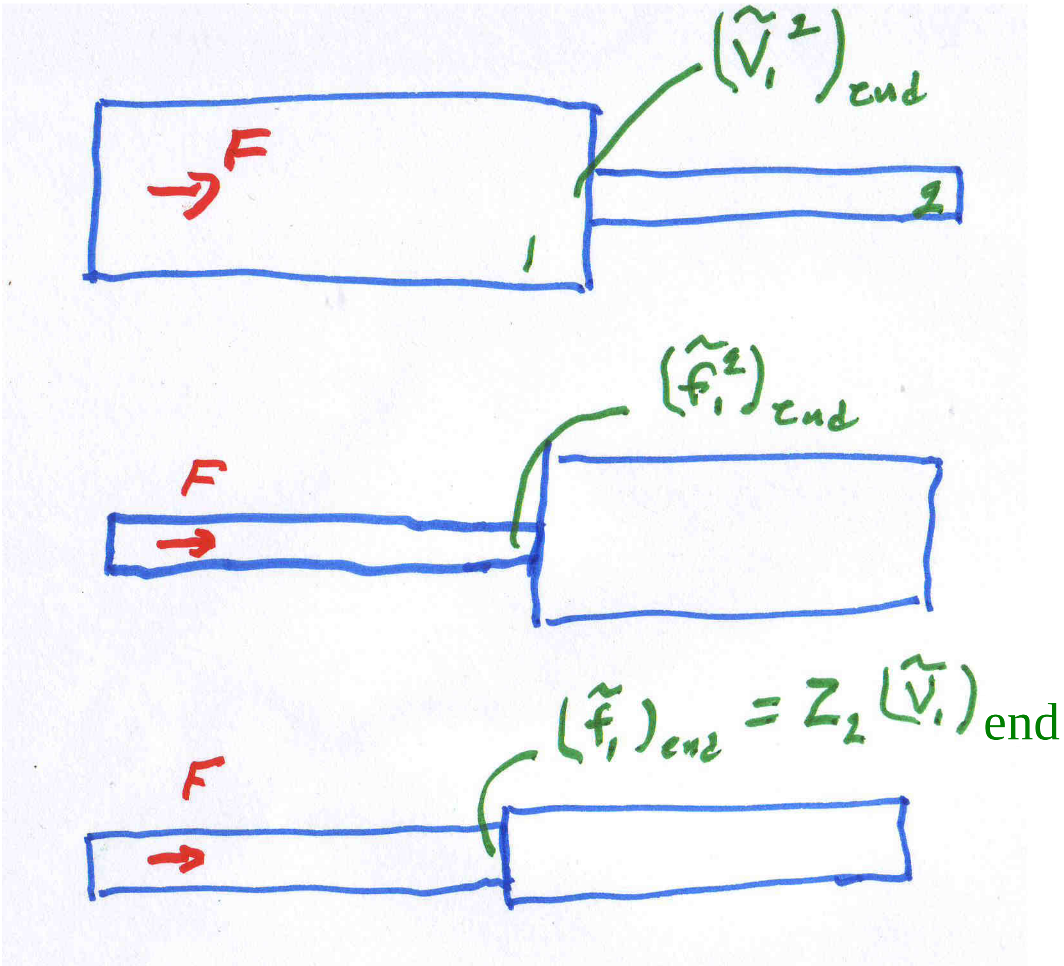
CLF is here for a rigid connector. It’s based on the assumption of “weak coupling”

One-Way procedures for CLFs…

- One junction at a time (compact support)
  - FRFs defined by elliptic equations, so, this cannot be exact
  - Doesn't work for 3 coupled oscillators (Woodhouse 1989)
  - Exact CLFs depend on damping, thus they depend on coupling damping at the next junction (Finnveden 1995)
- Define field in first element
  - Element has Finite Impedance / Finite Mobility , so, it cannot be an energy source

- SEA requires Weak Coupling
  - Errors when it's not Weak Coupling ?
    - Errors in dB ?
    - Errors in Physics -> errors in trends ?
  - When is it strong coupling ?
  - What can we do ?

Dynamic Coupling Strength / Strength of Connection



$$\left(\tilde{v}_1^2\right)_{end} \approx \frac{(e_1)_{end}}{\mu_1} \approx \frac{\hat{e}_1 n_1}{L_1 \mu_1} \approx \frac{\hat{e}_1}{\pi (\mu c_L)_1}$$

$$P_c^{(1,2)} = \mathrm{Re}(Z_2)\left(\tilde{v}_1^2\right)_{end} = \frac{1}{\zeta_2}\frac{\hat{e}_1}{\pi}$$

$$\left(\tilde{f}_1^2\right)_{end} \approx (EA)_1 (e_1)_{end} \approx (\mu c_L)_1 \frac{\hat{e}_1}{\pi}$$

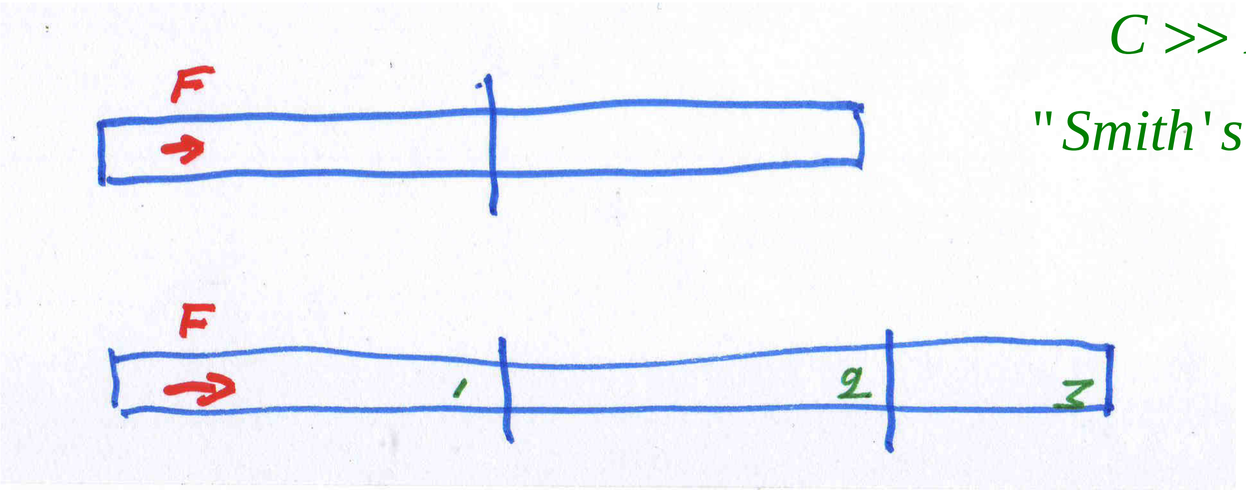
$$P_c^{(1,2)} = \mathrm{Re}(Y_2)\left(\tilde{f}_1^2\right)_{end} = \zeta_2 \frac{\hat{e}_1}{\pi}$$

$$P_c^{(1,2)} = \mathrm{Re}(Z_2)\left(\tilde{v}_1^2\right)_{end} = \frac{2}{\pi}\frac{\mathrm{Re}(\zeta_2)}{1+\zeta_2^2}\hat{e}_1$$

$$n_i = L_i / \pi c_{g,i} \qquad \zeta_2 = \frac{(\mu c_L)_2}{(\mu c_L)_1}$$

Vibroacoustic Reciprocity: $C_{1,2} = C_{2,1}$

Very strong coupling



$$C >> M_i,\ C >> M_j \Rightarrow \hat{e}_1 \approx \hat{e}_2$$

"Smith's strong coupling criterion"

Full Matrix:

$$\begin{bmatrix} M_1 + C^{1,2} + C^{1,3} & -C^{1,2} & -C^{1,3} \\ -C^{1,2} & M_2 + C^{1,2} + C^{2,3} & -C^{2,3} \\ -C^{1,3} & -C^{2,3} & M_3 + C^{2,3} + C^{1,3} \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \begin{bmatrix} \Pi_{in} \\ 0 \\ 0 \end{bmatrix}$$

"SEA gives the right answer for the wrong reason"
A.J. Keane

- Tunnelling is a mathematical artefact
  - When global modes dominates response
  - When spatial damping decay matters
    - Barbagallo *ISMA 2010:* $\eta L\omega / c_g > 1$

What is the Right Reason?

Impulse Response

$$M_1 \ddot{x}_1 + C_1 \dot{x}_1 + (K_1 + K_c) x_1 - K_c x_2 = 0$$

$$M_2 \ddot{x}_2 + C_2 \dot{x}_2 + (K_2 + K_c) x_2 - K_c x_1 = 0$$

$$x_1 = x_2 = \dot{x}_2 = 0, \quad \dot{x}_1 = 1/m_1 \quad \text{at } t = 0$$

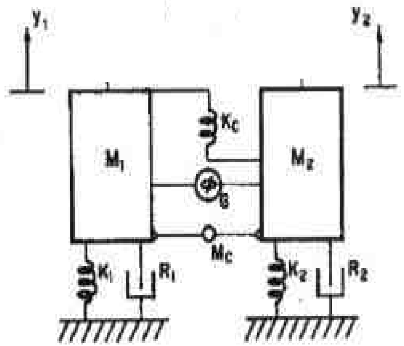


FIG 3.1
TWO LINEAR RESONATORS COUPLED BY SPRING, MASS, AND GYROSCOPIC ELEMENTS

$$x_2 = A_1 \left(\frac{1}{b_1} e^{-\alpha_1 t} \sin \Omega_1 t - \frac{1}{b_2} e^{-\alpha_2 t} \sin \Omega_2 t \right)$$

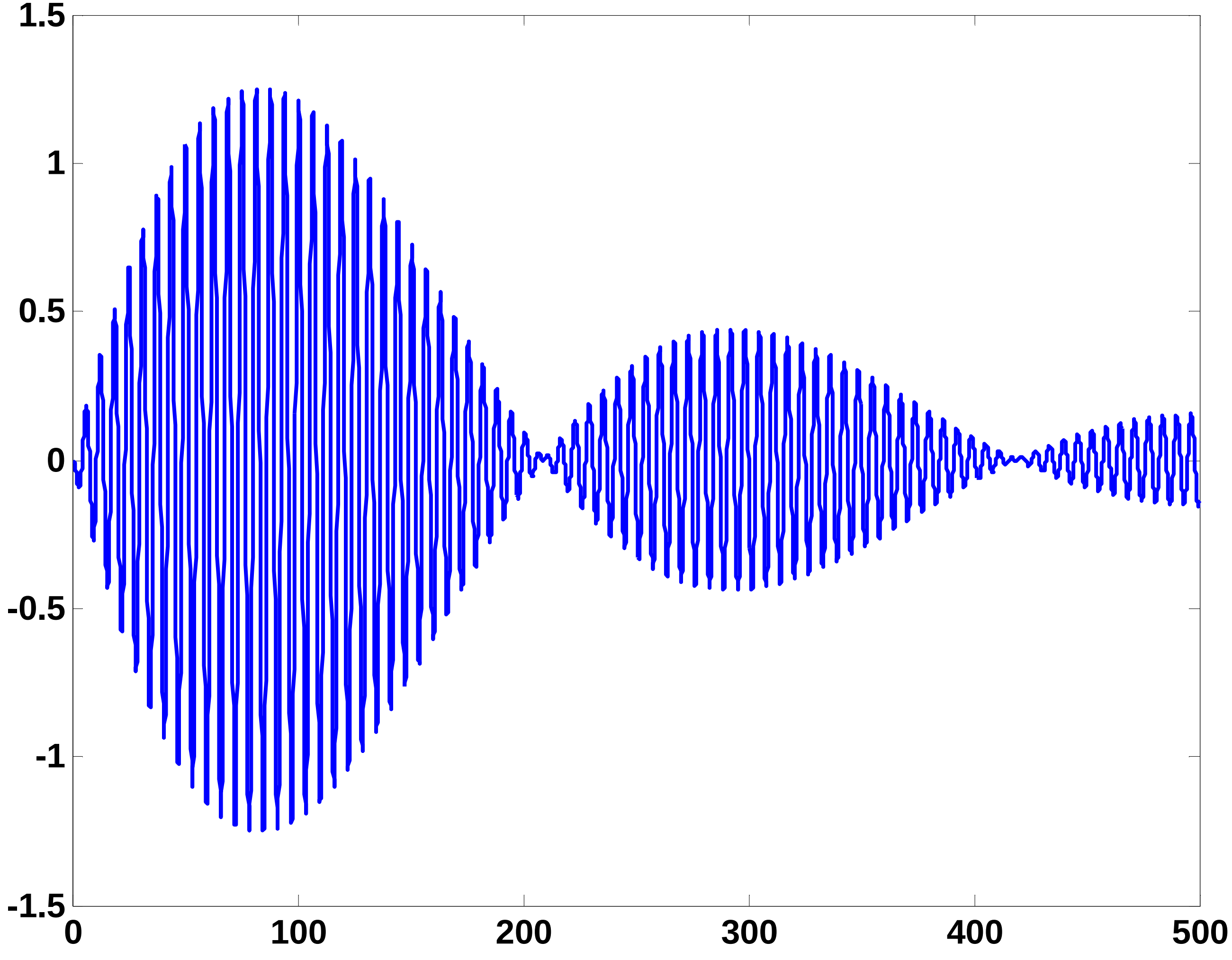
Wearing pushes coupled frequencies apart

$$\alpha_{1,2} = -\mathrm{Im} \sqrt{\tilde{\omega}_s^2 \pm \sqrt{\left(\tilde{\omega}_1^2 - \tilde{\omega}_2^2\right)^2 / 4 + \chi^2}}$$

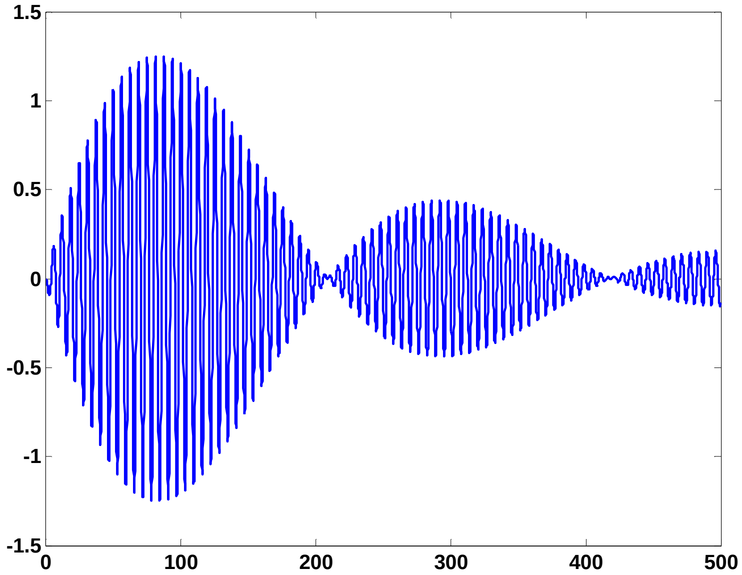
$$\Omega_{1,2} = \mathrm{Re} \sqrt{\tilde{\omega}_s^2 \pm \sqrt{\left(\tilde{\omega}_1^2 - \tilde{\omega}_2^2\right)^2 / 4 + \chi^2}}$$

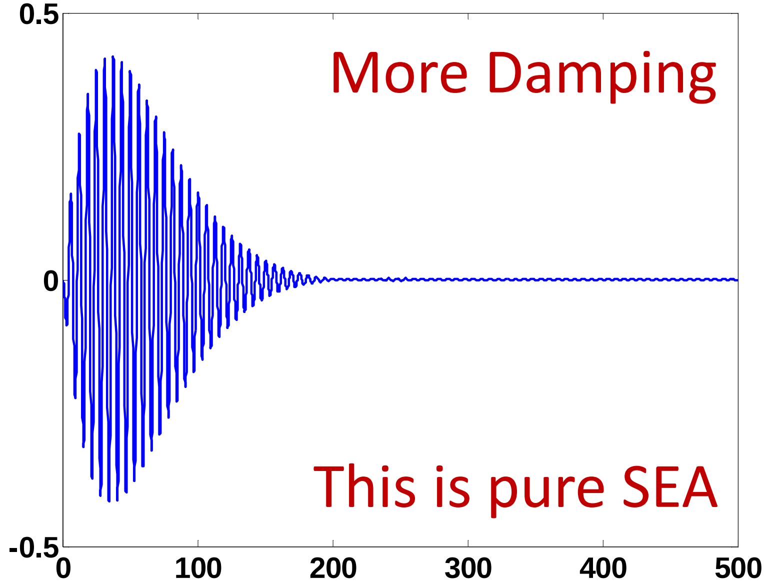
$$\tilde{\omega}_s = \sqrt{\left(\tilde{\omega}_1^2 + \tilde{\omega}_2^2\right)/2} \qquad \tilde{\omega}_i^2 = \left(\tilde{k}_i + k_c\right)/m_i = \omega_i^2 \left(1 - i\,\eta_i\right) \qquad \chi = k_c / \sqrt{m_1 m_2}$$

Response of Oscillator 2

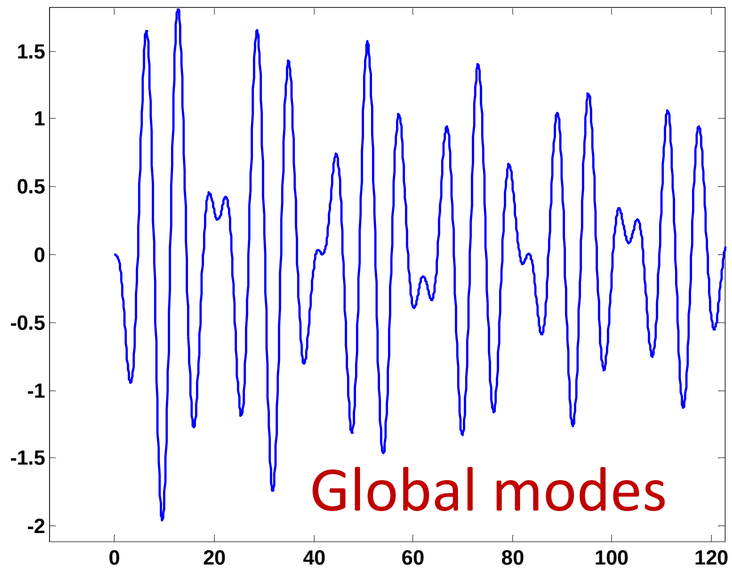


Response of Oscillator 2 ..





See also Lyon&DeJong
pp 52 (equal oscillators)



Stiffer
Coupling
spring

Failed to do a decent parallel to
Einstein (1905): Viscosity from
Brown’s motion … (fantastic!)

Short Time Average

Finnveden, *ISVR-TR*, 1997

Add one-way approx

Kinetic Energy, 2nd Oscillator

$$e_k = 2\,e^{-T} \sin^2 \kappa T$$

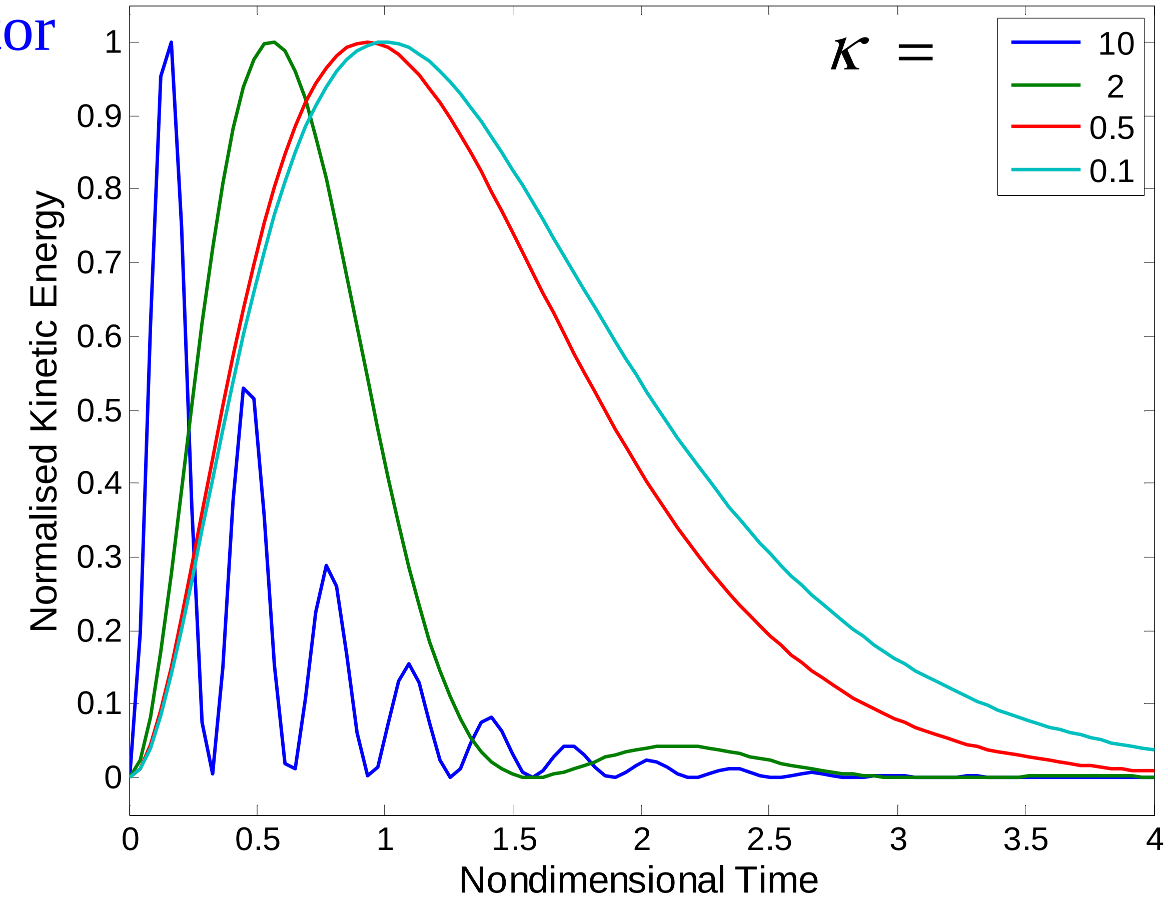
$$T = \eta\,\omega t;$$

$$\kappa = 2\sqrt{\delta + \gamma}$$

$$\delta = \left(\frac{\omega_1 - \omega_2}{\eta\,\omega}\right)^2$$

$$\gamma = \frac{k_c^2}{\eta^2\,\omega^4\,m_1\,m_2}$$

$$\eta = \eta_1 = \eta_2$$



Fahy & James (JSV -96) measured this for coupled plates. Response in plate 2 is dominated by mode-pairs: $\delta < 1$

Across an ensemble

$$\gamma < 1 \Rightarrow \kappa < 1$$

Steady State Energy Flow

$$\left(k_1 + k_c - \mathrm{i}\,\omega c_1 - \omega^2 m_1\right) U_1 - k_c\, U_2 = F_1 ,$$

$$-k_c\, U_1 + \left(k_2 + k_c - \mathrm{i}\,\omega c_2 - \omega^2 m_2\right) U_2 = 0$$

$$P_{in.1} = \mathrm{Re}\left(-i\,\omega\, u_1\, f^*\right) =$$

$$= \frac{\left(r_2^2 + 1\right) + \gamma}{\left(r_1\, r_2 - \gamma - 1\right)^2 + \left(r_1 + r_2\right)^2} \, \frac{\omega\, |f|^2}{m_1\, \Delta_1} ,$$

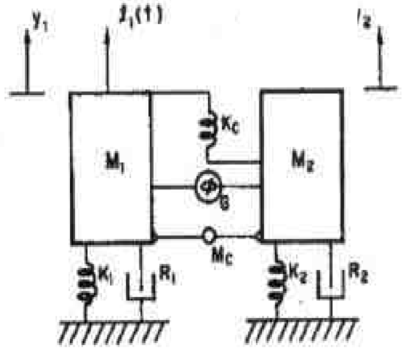


FIG 3.1
TWO LINEAR RESONATORS COUPLED BY SPRING, MASS, AND GYROSCOPIC ELEMENTS

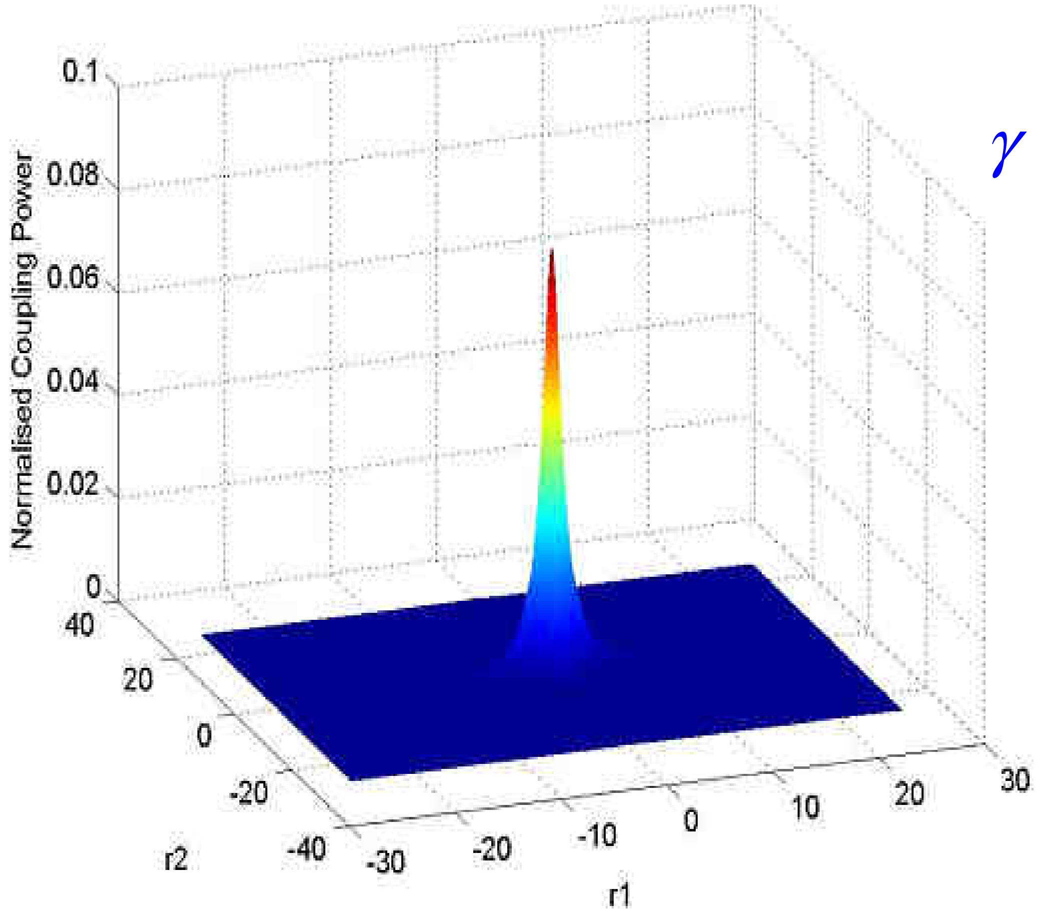
$$P_{coup}^{1.2} = \mathrm{Re}\left(-i\,\omega\, u_1\, k_c\left(u_1 - u_2\right)^*\right) =$$

$$= \frac{\gamma}{\left(r_1\, r_2 - \gamma - 1\right)^2 + \left(r_1 + r_2\right)^2} \, \frac{\omega\, |f|^2}{m_1\, \Delta_1}$$

$$r_i = \left(\omega_i^2 - \omega^2\right) / \left(\eta\, \omega^2\right)_i$$

$$\omega_i^2 = \left(k_i + k_c\right) / m_i$$

$$\gamma = \frac{k_c^2}{\eta_1\, \eta_2\, \omega^4\, m_1\, m_2}$$



$\gamma = 1 / 10$

Figure 1. Normalised coupling power, $P_{coup}^{1.2} / \left(\omega \left|f\right|^2 / m_1 \Delta_1\right)$, for $\gamma = 0.1$. The non-

$$r_i = \left(\omega_i^2 - \omega^2\right) / \left(\eta_i \, \omega^2\right) \approx 2\left(\omega_i - \omega\right) / \left(\eta_i \, \omega\right)$$

$\gamma = 10$

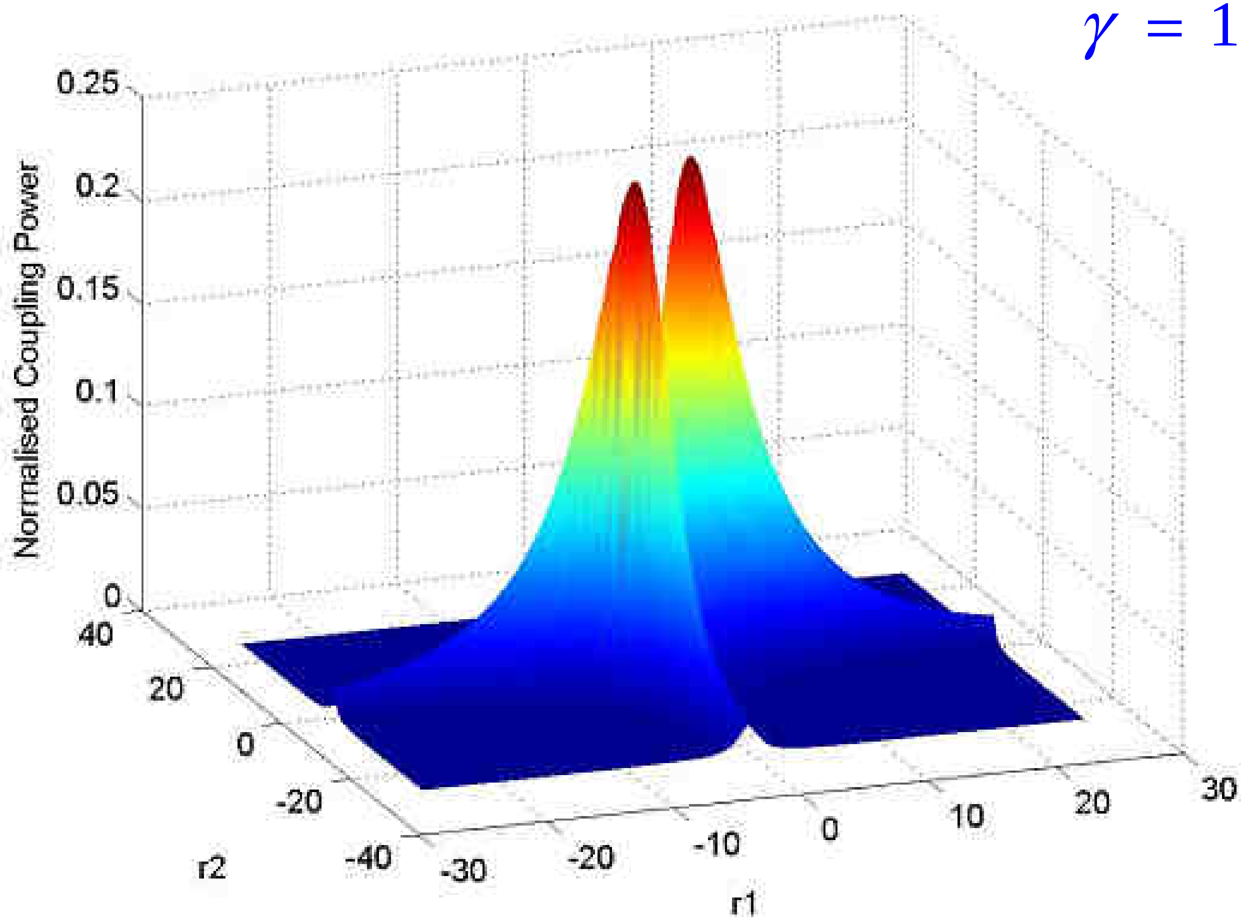
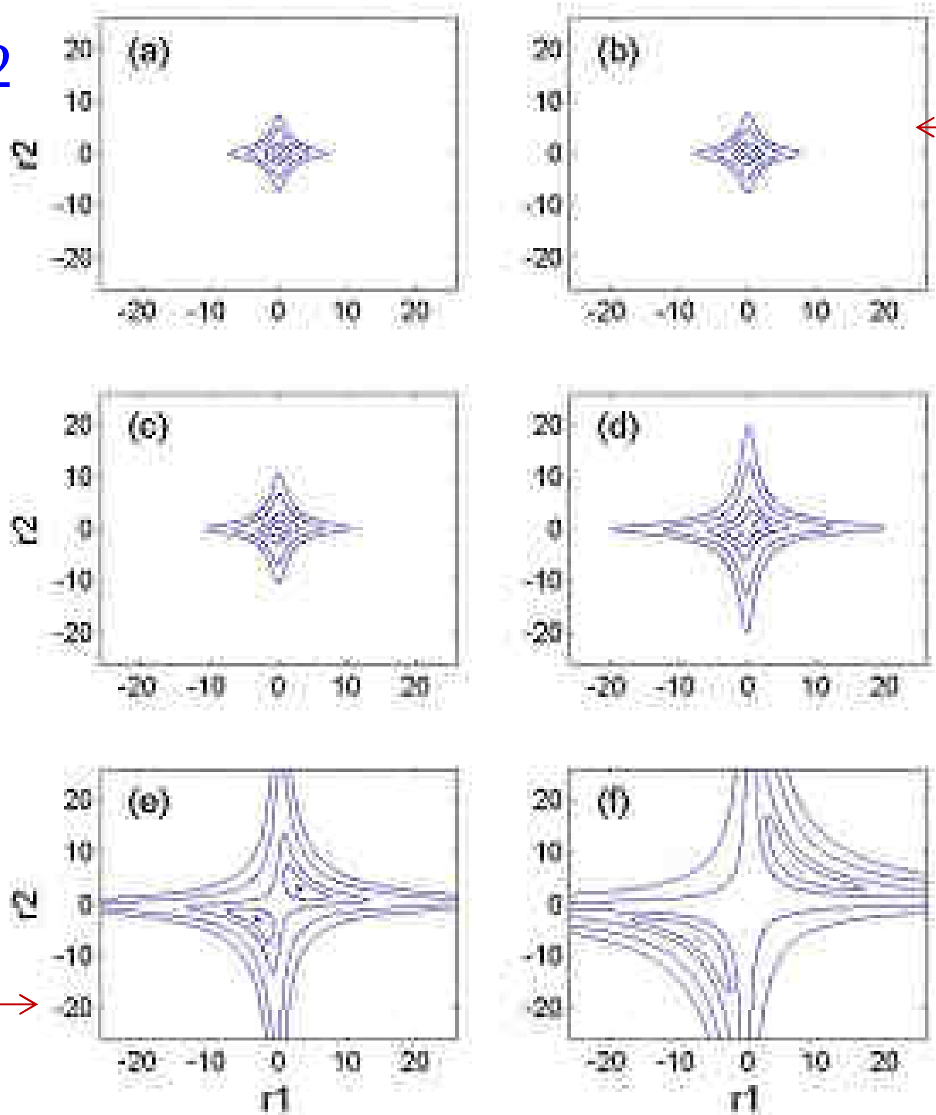
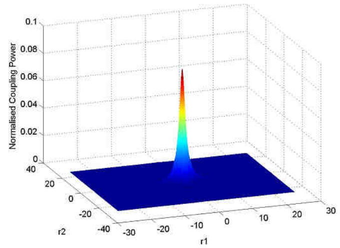


Figure 2. Normalised coupling power, $P_{coup}^{1,2} / \left(\omega \, |f|^2 / m_1 \, \Delta_1 \right)$, for $\gamma = 10$.

Normalised Coupling Power

$\gamma = 0.02$





$\gamma = 0.5$

$\gamma = 2$

$\gamma = 50$

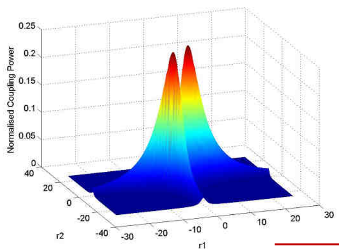


Figure 2. Normalised coupling power, $P^{1,2}_{coup}/\left(\omega |f|^2/m_1 \Delta_1\right)$, for $\gamma = 10$.

Wearing is not

apperant in response

if: $\gamma < 1$

γ - "Modal Interaction Strength"

Input Power (Input Mobility)

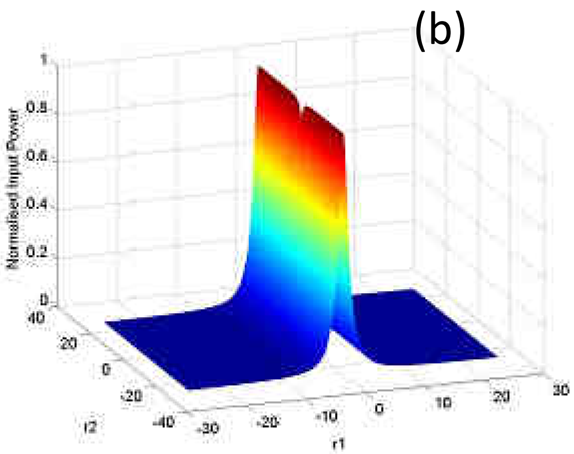


Figure 4. Normalised input power, $P_{in} / \left(\omega |f|^2 / m_1 \Delta_1 \right)$, for $\gamma = 0.1$.

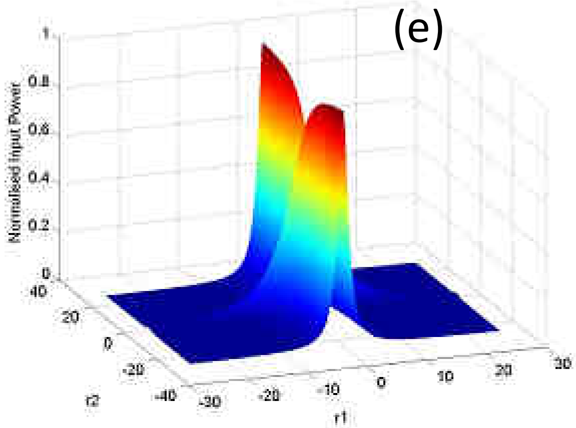
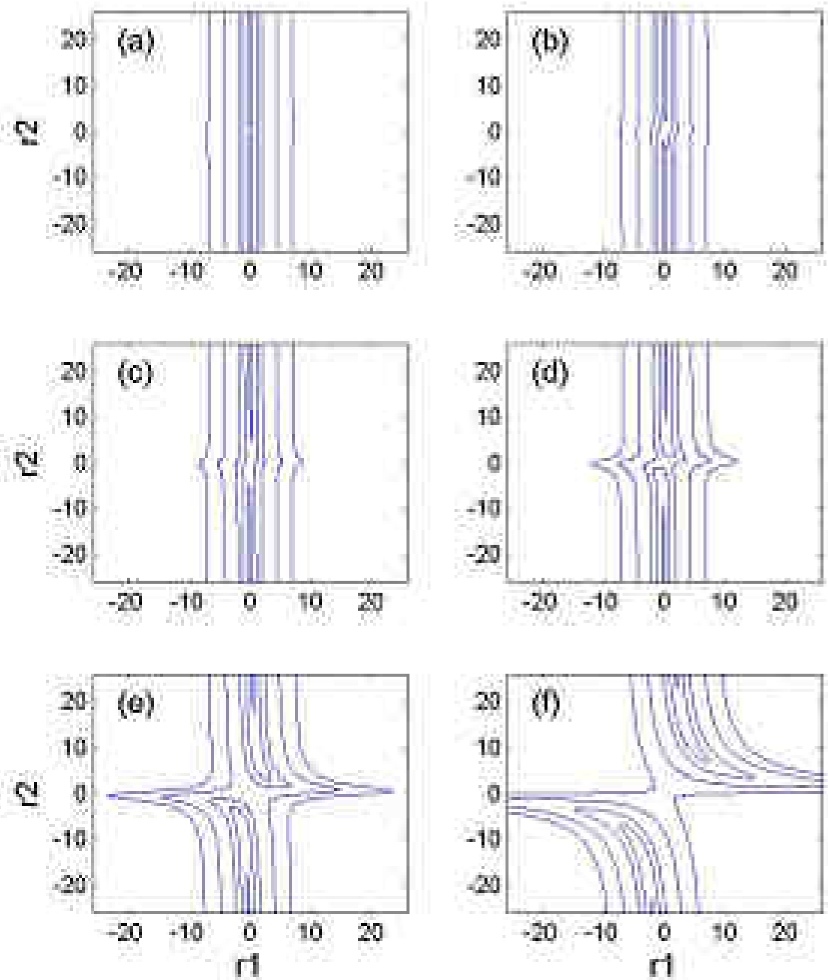
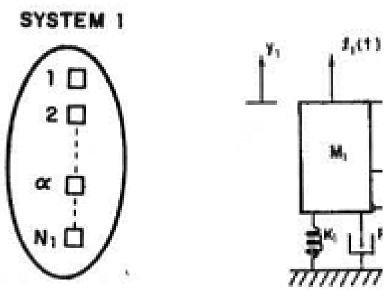


Figure 5. Normalised input power, $P_{in} / \left(\omega |f|^2 / m_1 \Delta_1 \right)$, for $\gamma = 10$.



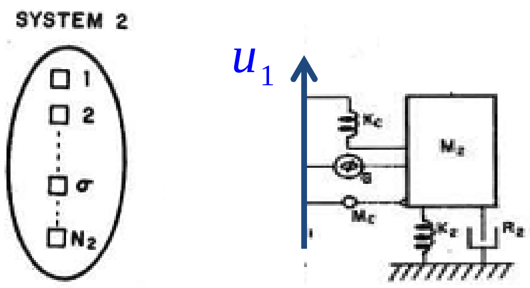
Langley – 89: “Coupling is weak if input mobility is unaffected by connected elements” true if: $\gamma < 1$

One-Way Approximation (standard SEA) ..



Motion of first Osc is a given quantity

$$\Rightarrow \langle \hat{e}_1 \rangle = \omega^2 M_1 \tilde{u}_1^2 / n_1$$



$$\Rightarrow \langle \hat{e}_2 \rangle \Rightarrow \langle P_{dissipated} \rangle = \langle P_{coup} \rangle$$

$$\langle P_{coup} \rangle = C_e \langle \hat{e}_1 \rangle$$

1) One resonance in band $\Delta\omega = \omega_u - \omega_l$

$$\Rightarrow n_i = 1/\Delta\omega$$

2) Replace $\omega \approx \omega_1 \approx \omega_2$, wherever possible

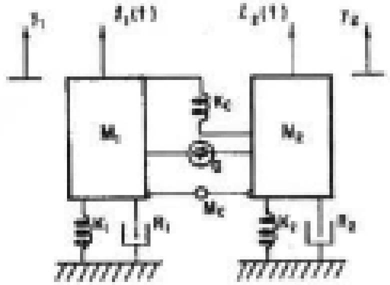
3) Use magic integral

$$C_e = \frac{\pi}{2} \frac{|\chi(\omega)|^2}{\omega^2 (\Delta\omega)^2}$$

$$\chi = \frac{(k_c + \mathrm{i}\,\omega g_c - \omega^2 m_c / 4)}{\sqrt{m_1 m_2}}$$

Scharton & Lyon JASA (1968)

One, out of very few, demonstrations of CPP based on fully coupled solutions



$$\left\langle P_c^{(1,2)} \right\rangle_\omega = \beta\left(\left\langle E_{k,1} \right\rangle_\omega - \left\langle E_{k,2} \right\rangle_\omega \right)$$

$$\mathscr{B} = \Big\{ \mu^2\left[\Delta_1 \omega_2^4 + \Delta_2 \omega_1^4 + \Delta_1\Delta_2(\Delta_1\omega_2^2 + \Delta_2\omega_1^2)\right] + (\gamma^2 + 2\mu\kappa)(\Delta_1\omega_2^2 + \Delta_2\omega_1^2) + \kappa^2(\Delta_1 + \Delta_2) \Big\} \times \Big\{ (1-\mu^2)\left[(\omega_1^2 - \omega_2^2)^2 + (\Delta_1 + \Delta_2)(\Delta_1\omega_2^2 + \Delta_2\omega_1^2)\right] \Big\}^{-1}.$$

$$\Delta_i = R_i / M_i^{'}; \quad \omega_i^2 = (K_i + K_c)/M_i^{'}; \quad M_i^{'} = M_i + M_c/4;$$

$$\kappa^2 = K_c^2 / M_1^{'} M_2^{'}; \quad \gamma = G\, M_1^{'} M_2^{'}; \quad \mu = M_c\,\lambda/4; \quad \lambda = M_1^{'}/M_2^{'}$$

Frequency averaged response of Two Coupled Oscillators

- The Power flow is proportional to the difference of the Oscillators' Energy
- The constant of proportionality is positive definite
- The constant of proportionality is Symmetric in system parameters
    - $\rightarrow$ CPP

- If one oscillator is excited, the energy of the second oscillator cannot be greater than that of the first oscillator
- However, the constant of proportionality depends on oscillator damping.
    - Hence, CPP cannot be exact for three oscillators
        - As been proven (Woodhouse 1981)

Ensemble Averages

- Lyon 1975, Mace and Ji 2007

$(\omega_1 - \omega_2)$ is rectangle distributed.

$$\text{pdf}\,(\omega_1 - \omega_2) = \begin{cases} 1/\Delta\omega & \text{in band} \\ 0 & \text{otherwise} \end{cases}$$

$\Delta\omega$ is wide enough

Replace $\omega \approx \omega_1 \approx \omega_2$, wherever possible

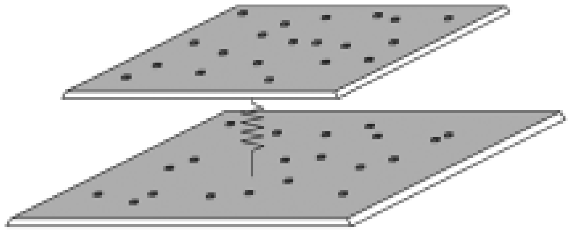
$$C = C_e \big/ \sqrt{1+\gamma};$$

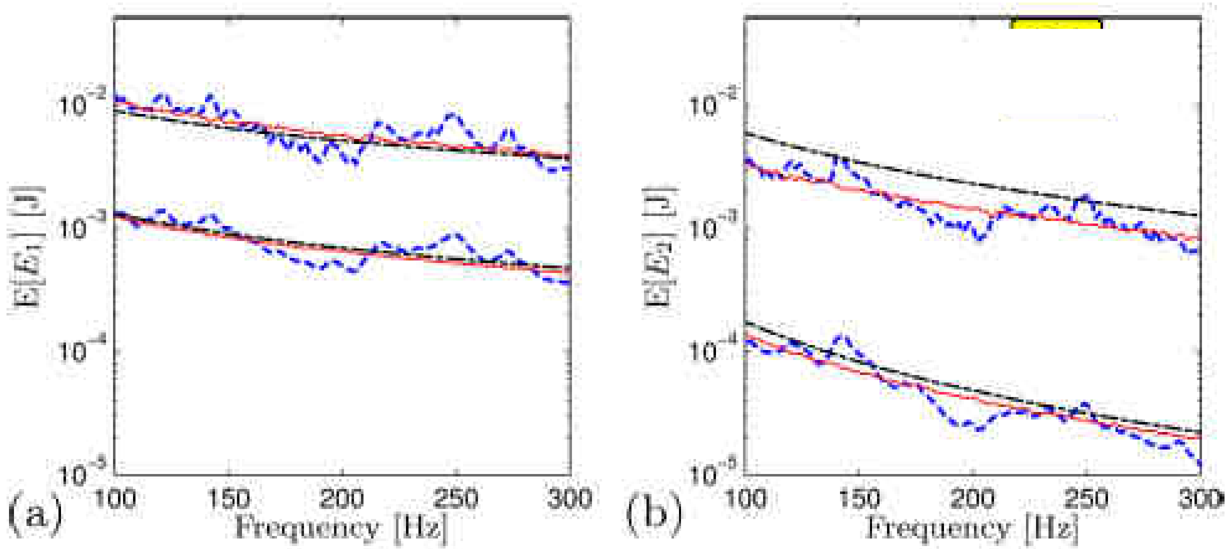
$$\gamma = \frac{2}{\pi}\frac{C_e}{M_1 M_2} = \frac{2}{\pi}\frac{\pi}{2}\frac{\left|\chi(\omega_n)\right|^2 n_1 n_2}{\omega^2 (\eta n \omega)_1 (\eta n \omega)_2} = \frac{\left|\chi(\omega_n)\right|^2}{\eta_1 \eta_2 \omega^4}$$

If $\gamma < 1$, the one-way approach is OK, otherwise the Connectivity depends on damping (and thus on coupling damping)

Reynders 2014

Upper curves $\eta = 0.01$; Lower curves $\eta = 0.001$





Seems as the values of γ is incorrect in the article

Black dash-dot: 'SEA'

- Weaker losses -> Stronger modal interaction strength
  - SEA over predicts transmission
  - But not much

SEA of Two-element Structure

(One more, out of very very few, fully coupled demonstrations)

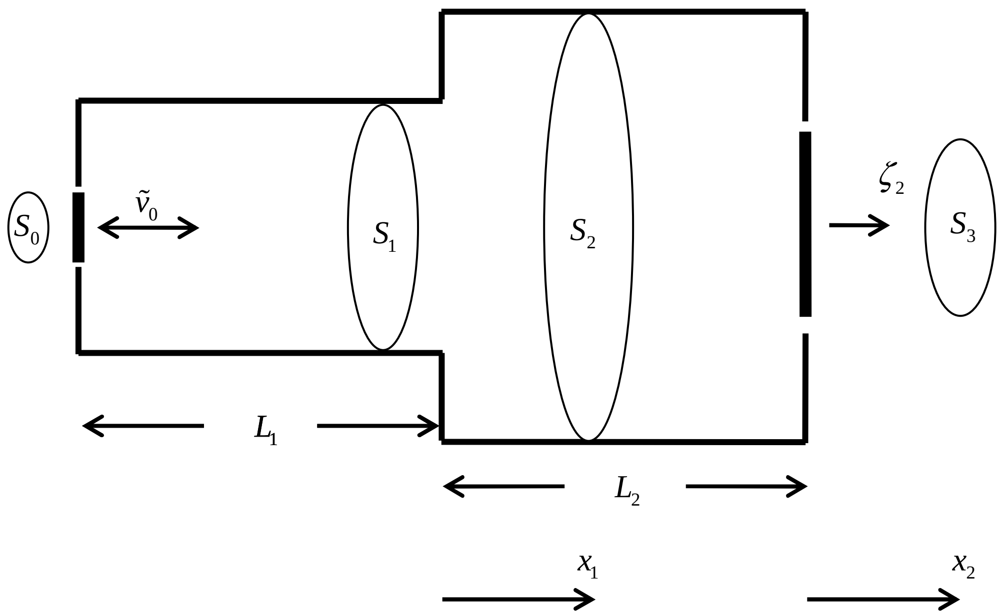


Figure 3- 9. Two-element structure.

$$M_i = \frac{\eta_i k_0 L_i}{\pi}; \quad C_e = \frac{2}{\pi} \frac{\mathrm{Re}(\zeta_2)}{1 + |\zeta_2|^2}$$

Energies and Energy Flows

$$e_p = \frac{S_1 \left|\tilde{p}_1\right|^2}{2\rho_0 c} = \frac{\Pi_{in}}{2c} \frac{\left|\sin\left(kx+\psi\right)\right|^2}{\left|\cos\left(kL-\psi\right)\right|^2},$$

$$e_k = \frac{S_1 \rho_0 \left|\tilde{v}_1\right|^2}{2} == \frac{\Pi_{in}}{2c} \frac{\left|\cos\left(kx+\psi\right)\right|^2}{\left|\cos\left(kL-\psi\right)\right|^2},$$

$$\Pi_{in} = \rho_0 c \left|\tilde{v}_0\right|^2 S_0^2 / S_1 .$$

$$P_{in} = S_1 \operatorname{Re}\left(\tilde{p}_1\left(x=-L\right)\tilde{v}_1^*\left(x=-L\right)\right) = \Pi_{in} \operatorname{Re}\left(-\mathrm{i}\tan\left(kL-\psi\right)\right),$$

$$P_{tr} = S_1 \operatorname{Re}\left(\tilde{p}_1\left(x=0\right)\tilde{v}_1^*\left(x=0\right)\right) = \Pi_{in} \frac{\sinh\left(-2\operatorname{Im}\left(\psi\right)\right)}{2\left|\cos\left(kL-\psi\right)\right|^2},$$

$$\psi = \operatorname{atan}\left(\zeta / \mathrm{i}\right)$$

Ensemble averages

Frequency Dependance: $\omega L/c$

If L or c are (rectangel distributed)
Random Variables

1 element:

Ensemble Av. = Frequency Av.

$$\omega(\eta+\eta_c)\langle E_1\rangle_{k_oL}=\langle P_{in}\rangle_{k_oL}=\Pi_{in}$$

Exact Ensemble Averages
Two Elements

Random elements: $$(k_0L)_i = \left\langle (k_0L)_i \right\rangle + \mathrm{R}[-\pi/2, \pi/2]$$

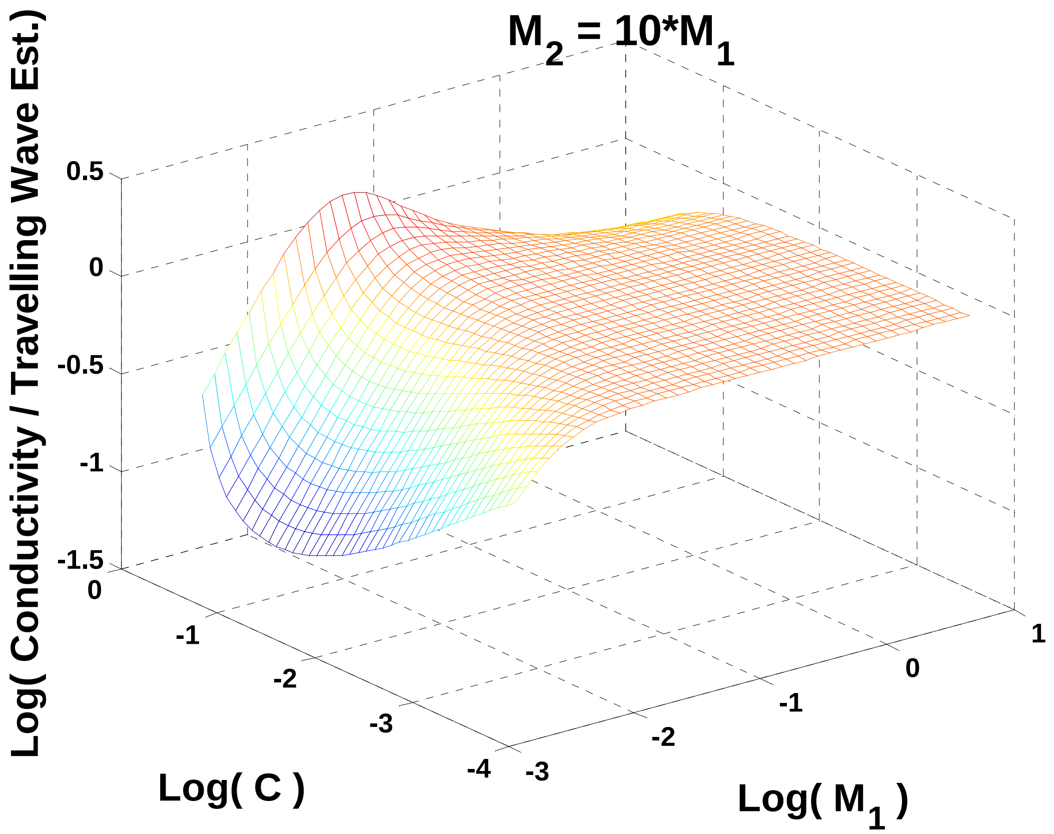
$$\left\langle P_{coup}^{1,2} \right\rangle_{(kL)_1,(kL)_2} = C_e \left(\left\langle \hat{e}_1 \right\rangle_{(kL)_1,(kL)_2} - \left\langle \hat{e}_2 \right\rangle_{(kL)_1,(kL)_2} \right)$$

$$C_e = \frac{C}{Q - C/M_1 - C/M_2}, \quad C = \frac{2}{\pi} \frac{\mathrm{Re}(\zeta)}{1+|\zeta|^2},$$

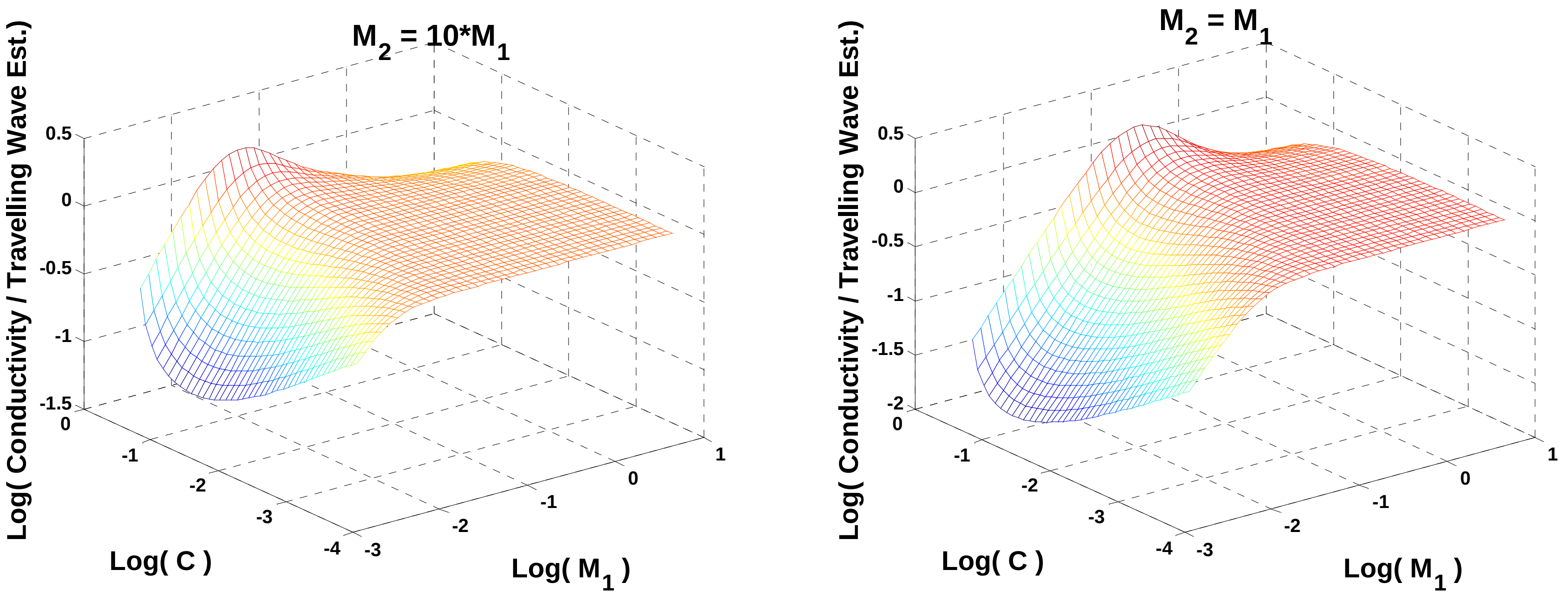
$$Q = \sqrt{1 + \frac{2\pi C}{\tanh(\pi M_1)\tanh(\pi M_2)} + \left(\frac{\pi C}{\tanh(\pi M_1)} \right)^2 + \left(\frac{\pi C}{\tanh(\pi M_2)} \right)^2 - (\pi C)^2}$$

- Proved CPP for ensemble averages
- If $\gamma < 1$, *the* one-way approach is OK, otherwise the Connectivity depends on damping (and thus on coupling damping)

Conductivity for Exact Ensemble normalised with Travelling Wave Estimate



Conductivity for Exact Ensemble normalised with Travelling Wave Estimate



Errors are not large if: $2C / (\pi M_1 M_2) < 1$

3 modes in a 1/3-octave band η=0.01 -> M=0.12

Comments on Coupling strength

- Dynamic coupling / Connection strength
  - Critical to get right, when using a one-way method, and, for coupling conditions, when using uncoupled modes
- Smith's criterion $C << M_i$
  - Describes the character of SEA solutions
  - Does not validate an SEA model
  - If it is large, SEA might give the right answer for the wrong reason
- Modal interaction strength
  - Defines if response is given by local or global modes
  - If it is large, CLFs are smaller and might depend on damping
    - Experimentally verified for a ship structure (Nilsson 1978)
    - Critical as noise control measures are incorrectly predicted
  - Might validate Langley's weak coupling criterion: Point mobility for uncoupled and coupled elements are equal
    - If so, we can measure point mobilities and check
  - Might be observed from the impulse response of a connected element