DMA – Università di Roma La Sapienza
Lab. di Vibrazione e Acustica Strutturale

Complex Envelope Vectorization for the solution of mid-high frequency acoustic problems

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Presentation layout

- Low frequency mapping of HF vibration (CEDA)
- CEV approach and governing equations
- Under-sampling
- Limits of CEV and remarks
- Application of CEV to benchmarks
    - Test cases and results
- Application of CEV to Boundary Element Formulation
    - Test cases and results
- A reference model for future developments (WKB)
- Conclusions

LOW FREQUENCY MAPPING OF HF VIBRATIONS (CEDA)*

Assuming a forcing term of radiant frequency ω, the equation of motion for a one-dimensional undamped structure is:

$$\mathbf{L}[u] + m\omega^2 u = f$$

The **complex envelope displacement** is defined as follows:

$$\overleftarrow{u} = \mathbf{E}[u] = (u + j\tilde{u})e^{-jk_0 x}$$

E: envelope operator

$\tilde{u}$ is the Hilbert Transform of u, i.e. $\tilde{u}(x) = \int_{-\infty}^{+\infty} \frac{u(\xi)}{\pi(x-\xi)} d\xi$

and $(u + j\,\tilde{u})$ is the analytical displacement.

*Carcaterra and Sestieri: Complex envelope displacement analysis: a quasi-static approach to vibrations. JSV, vol. 201(2), 1997.

LOW FREQUENCY MAPPING OF HF VIBRATIONS (cont'd)

By applying the envelope operator to the motion equation and expressing the physical displacement in terms of the complex envelope one obtains:

$$\overleftarrow{\mathbf{L}}[\overleftarrow{u}] + m\omega^2 \overleftarrow{u} = \overleftarrow{f} = (f + j\,\overleftarrow{f})e^{-jk_0 x}$$

CEDA equation

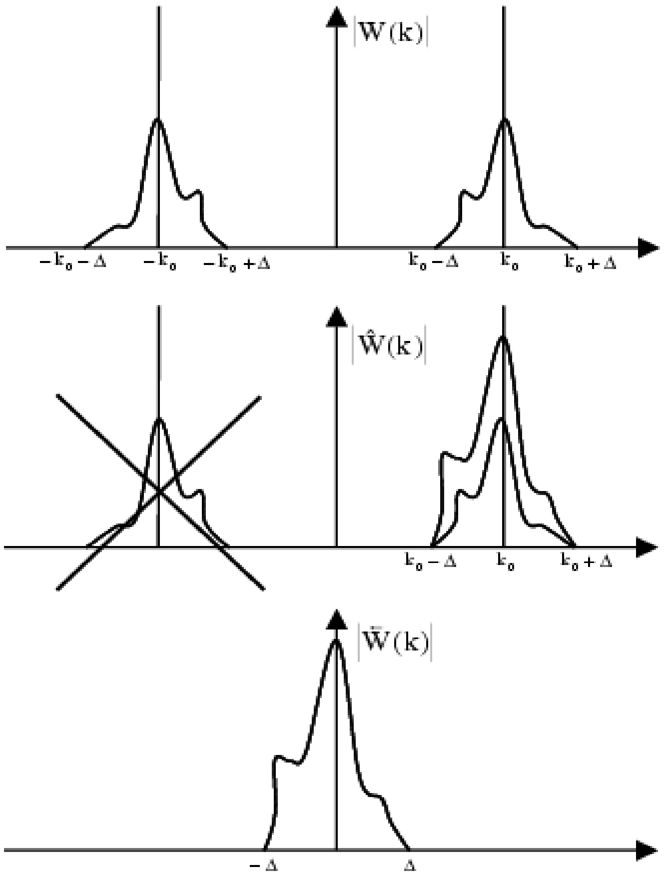
being $\overleftarrow{\mathbf{L}} = \mathbf{E}\;\mathbf{L}\;\mathbf{E}^{-1}$

$\overleftarrow{u}$ admits an inverse which is given by:

$$u = 2\,\mathrm{Re}\{\overleftarrow{u}\; e^{j k_0 x}\}$$

So, once the CEDA equation is solved, one can reconstruct the physical displacement

CEDA LAY-OUT IN THE WAVENUMBER DOMAIN

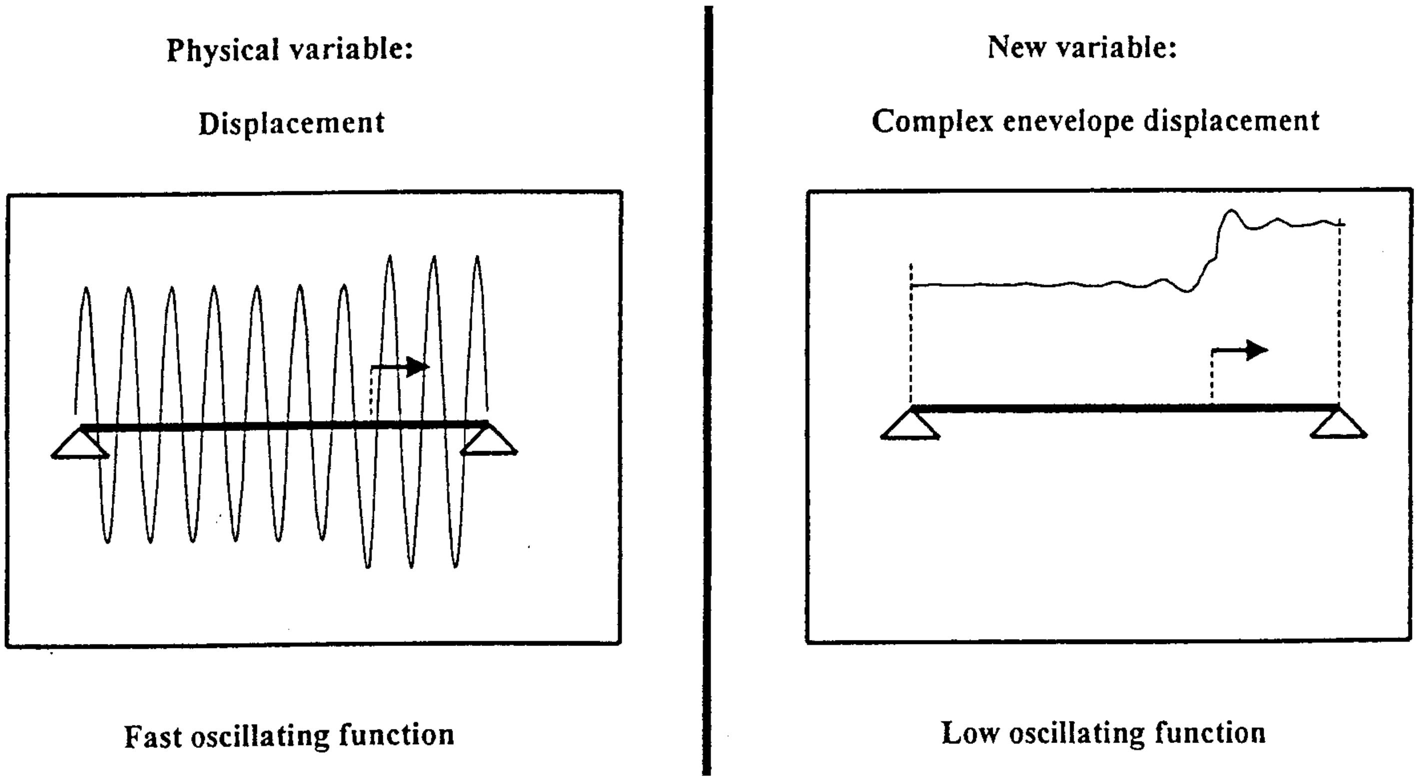


Physical signal spectrum

Analytic signal spectrum

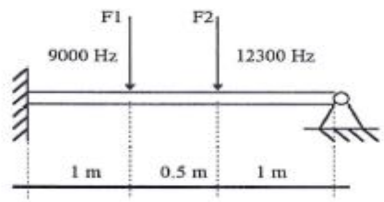
Shifted signal spectrum
(complex envelope)

LOW FREQUENCY MAPPING OF HF VIBRATIONS (CEDA)*



When computing the complex envelope displacement we can recover the physical displacement by the inverse transformation

LOW FREQUENCY MAPPING OF HF VIBRATIONS (CEDA)
Example



For the exact solution 100 points used against 16 points in CEDA

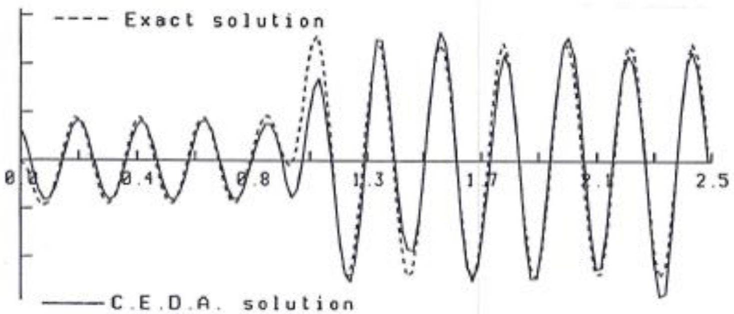
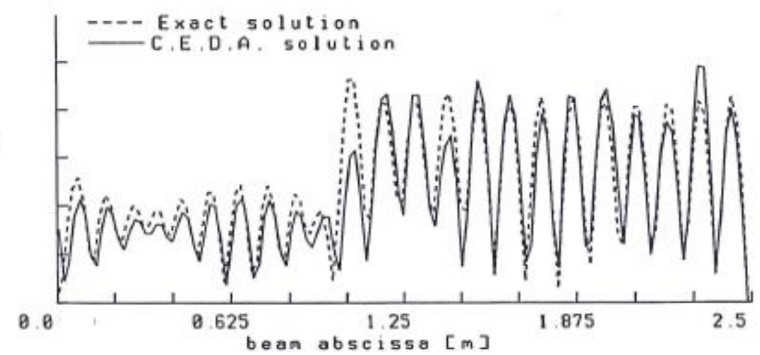


Fig.8



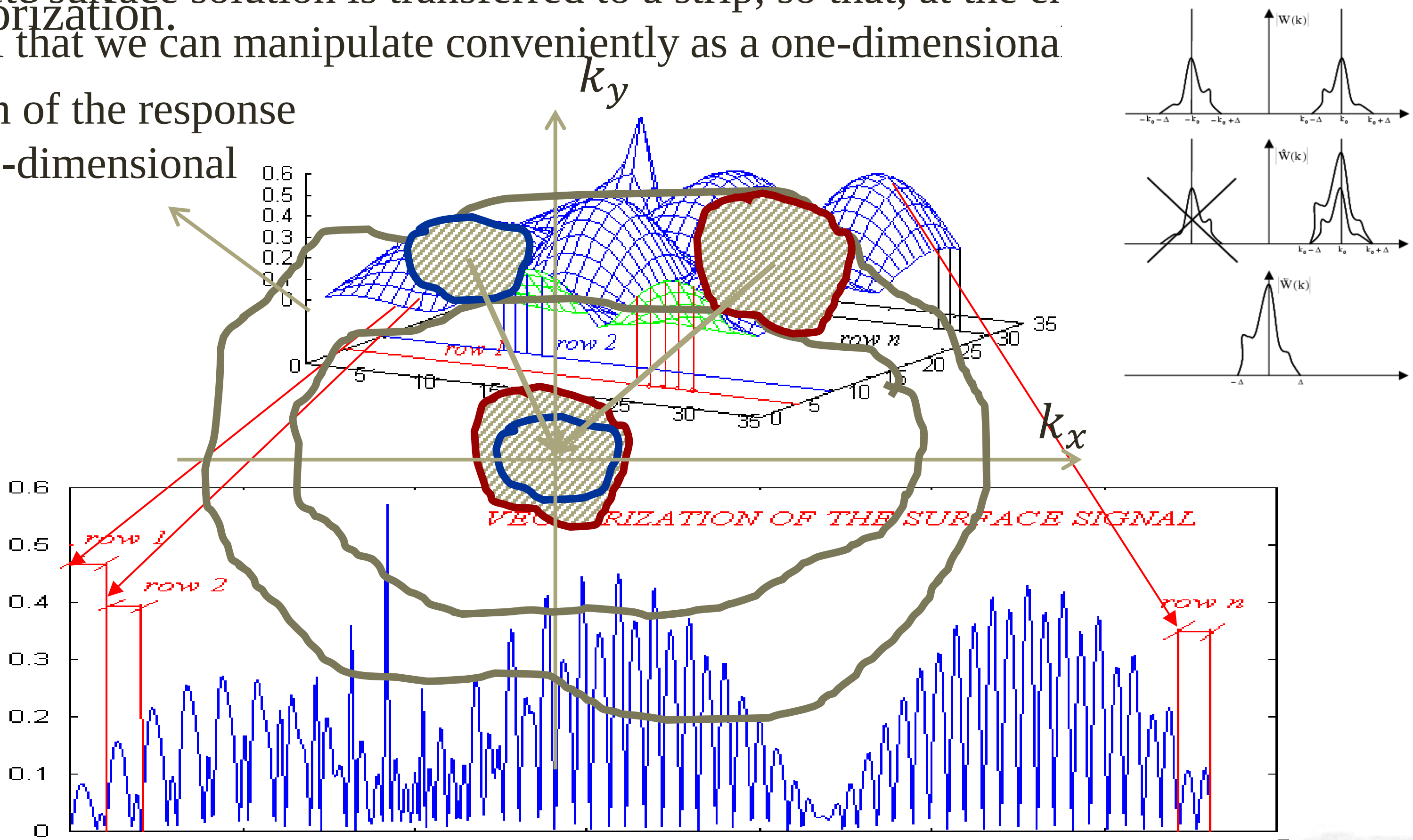
CEV: VECTORIZATION PROCESS

CEV: VECTORIZATION PROCESS

Because of the difficulties to extend CEDA to multi-dimensional and vibro-acoustic systems … another approach was proposed: **CEV** - Complex Envelope Vectorization.

This slide shows schematically the vectorization procedure: each row (or column) of the discrete surface solution is transferred to a strip, so that, at the end, we have a new discrete signal that we can manipulate conveniently as a one-dimensiona

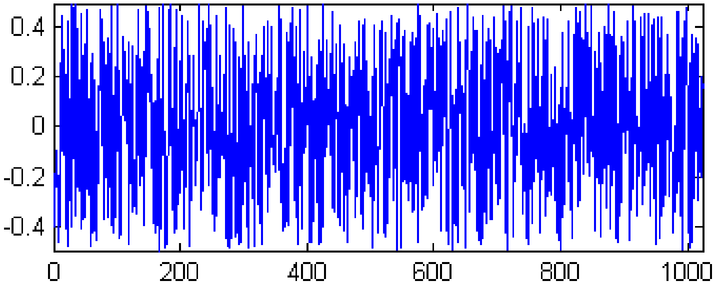
Spectrum of the response for a two-dimensional system



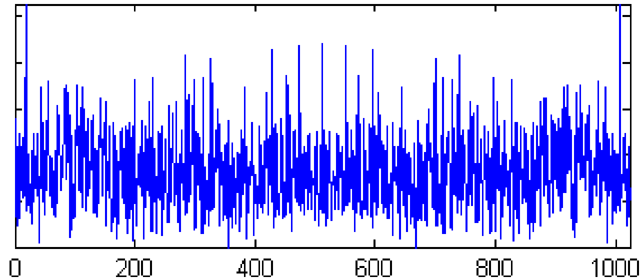
CEV VECTORIZATION PROCEDURE

Note: in the above vectorization procedure the spectrum of the response signal is not anymore concentrated around a single wavenumber k_0 but rather spreads into the whole wavenumber domain

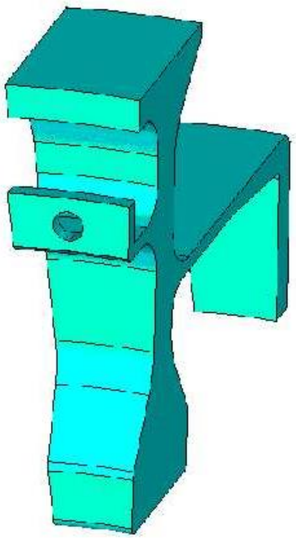
Vectorized signal

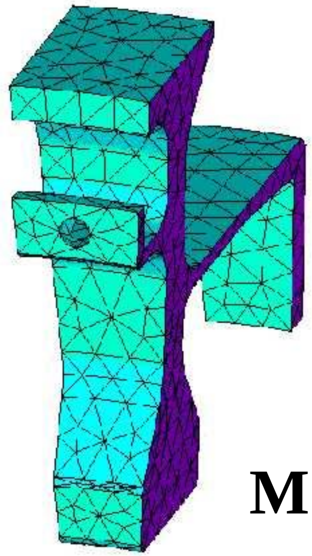


Fourier Transform



LOW FREQUENCY MAPPING OF HF VIBRATIONS (CEV)*





FEM

Stiffness matrix **K** and mass matrix **M**

$$\mathbf{M}\ddot{\mathbf{u}}+\mathbf{K}\mathbf{u}=\mathbf{p}$$

If the excitation is harmonic with frequency ω, one has

$$(-\omega^2 \mathbf{M} + \mathbf{K})\,\mathbf{u} = \mathbf{p} \quad \Rightarrow \quad \mathbf{A}\mathbf{u}=\mathbf{p}$$

* Giannini, Carcaterra, Sestieri. High frequency vibration analysis by CEV. *JASA*, 121(6), 2007

CEV (cont'd)

In CEV a set of complex envelope signals $\overleftarrow{\mathbf{u}}^{(r)}$ and $\overleftarrow{\mathbf{p}}^{(r)}$ is produced from **u**

From $\boxed{\mathbf{A}\mathbf{u} = \mathbf{p}}$ → ? → $\boxed{\overleftarrow{\mathbf{A}}^{(r)}\overleftarrow{\mathbf{u}}^{(r)} = \overleftarrow{\mathbf{p}}^{(r)}}$

(CEV equation)

LOW FREQUENCY MAPPING OF HF VIBRATIONS (CEV)*

$$(-\omega^2\mathbf{M}+\mathbf{K})\mathbf{u}=\mathbf{p} \quad \Longrightarrow \quad \mathbf{A}\mathbf{u}=\mathbf{p}$$

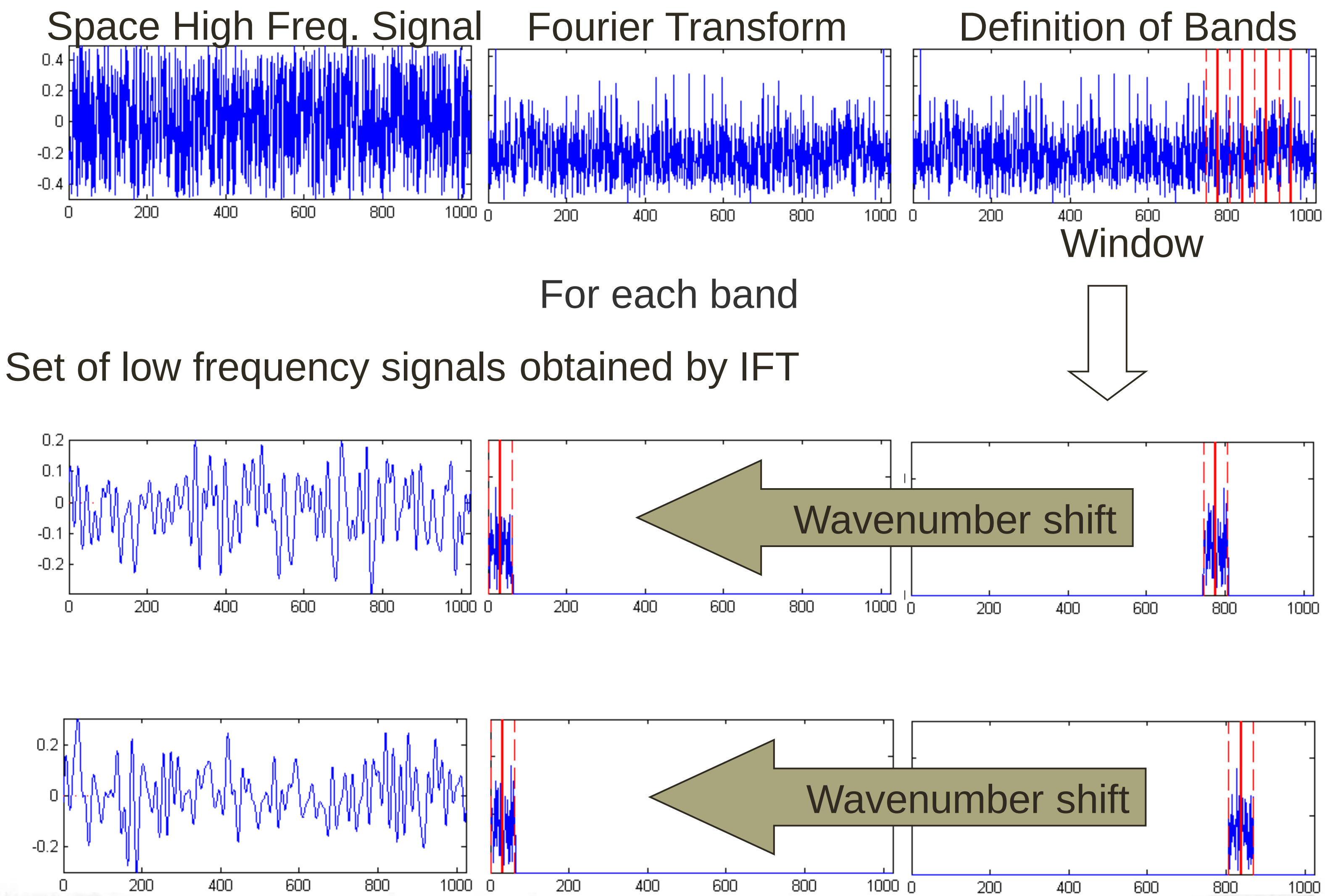
u and **p** can be viewed as sampled values of continuous signals *u(s)* and *p(s)* .

Because of the vectorized spread in the wavenumber domain, in CEV two sets of complex envelope signals $\overleftarrow{u}^{(r)}(s)$ ($p^{(r)}(s)$) are produced from $u(s)$ ($p(s)$), each one characterized by a narrow wavenumber spectrum.

$$u(s) \quad \Rightarrow \quad \overleftarrow{u}^{(r)}(s) \quad (r=1,2,\ldots..,P)$$

This is obtained by filtering the spectrum $U(k)$ of $u(s)$ by a set of narrow spectral windows $W^{(r)}(k)$ and shifted by the quantity k_r towards the wave number origin to produce $\overleftarrow{U}^{(r)}$.

The Envelop operator for high dimensional problems



LOW FREQUENCY MAPPING OF HF VIBRATIONS (CEV) (cont'd)

The *rth* complex signal $\overleftarrow{u}^{(r)}$ is obtained by inverse Fourier Transform

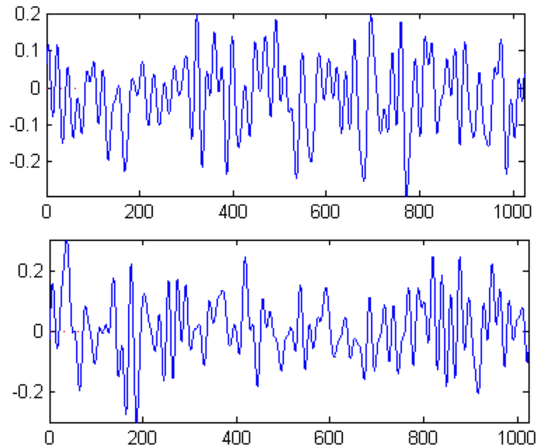
$$\overleftarrow{u}^{(r)}(s)=F^{-1}\{\overleftarrow{U}^{(r)}(k)\}=F^{-1}\{U(k+k_r)W^{(r)}(k+k_r)\}$$

and ….

The operation is reversible

(provided that the whole set of windowed signals are considered)

Set of low wavenumber signals



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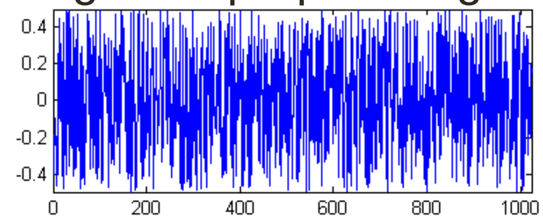
High Freq. Space Signal

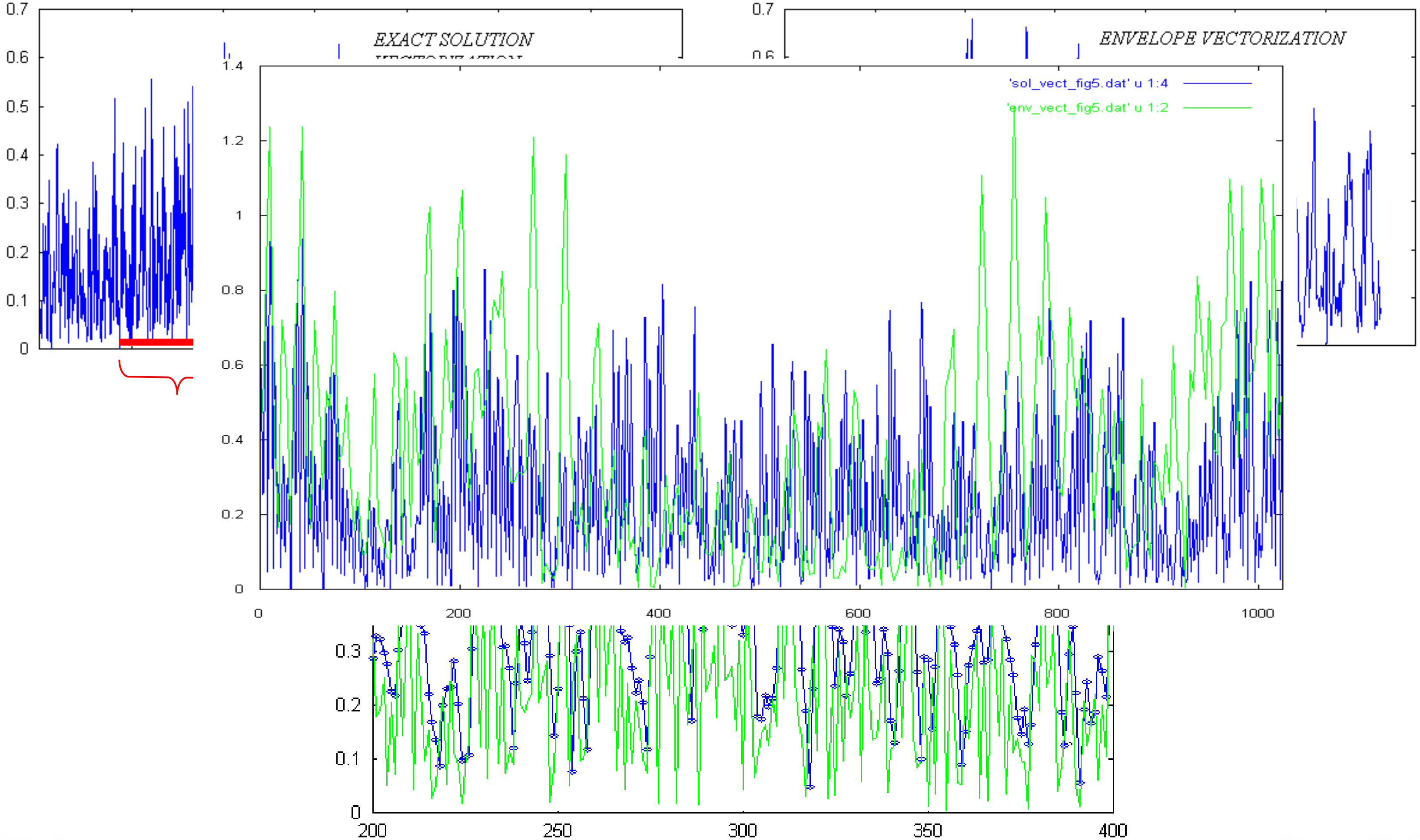


$$u(s) \quad \Rightarrow \quad \overleftarrow{u}^{(r)}(s) \quad (r = 1, 2, \ldots.., P)$$

$$\overleftarrow{u}^{(r)}(s) = F^{-1}\{\overleftarrow{U}^{(r)}(k)\} = F^{-1}\{U(k+k_r)W^{(r)}(k+k_r)\}$$

$$u(s) = 2\,\mathrm{Re}\{\sum_{r=1}^{P} \overleftarrow{u}^{(r)}(s)\; e^{jk_r s}\}$$

NUMERICAL RESULTS (34x34)



How to build $\overleftarrow{\mathbf{A}}$ ($\overleftarrow{\mathbf{M}}$, $\overleftarrow{\mathbf{K}}$) from the physical matrix A (M, K)

The explicit form of the motion equation $\mathbf{Au} = \mathbf{p}$ is: $\sum_k A_{hk} u_k = p_h$

It can be turned into continuous form by using $p(s)$ and $u(s)$ to obtain:

$$\int_I a(s,\sigma) u(\sigma) d\sigma = p(s)$$

Assume the kernel to have the form: *a* (*s* ,*σ*) = *a* (*s* - *σ*).

By *i*) Fourier transforming the motion equation, *ii*) applying the window $W^{(r)}$, and *iii*) shifting the result toward the origin by k_r:

$$A(k+k_r) W^{(r)}(k+k_r)\, U(k+k_r) = P(k+k_r) W^{(r)}(k+k_r)$$

$$\left[F^{-1}\right] \qquad \text{(Inverse Fourier Transform)}$$

$$\int_I \overleftarrow{a}^{(r)}(s-\sigma) e^{-jk_r(s-\sigma)}\, u^{(r)}(\sigma)\, d\sigma = \overleftarrow{p}^{(r)}(\sigma)$$

How to build $\overleftarrow{\mathbf{A}}$ ($\overleftarrow{\mathbf{M}}$, $\overleftarrow{\mathbf{K}}$) from the physical matrix A (M, K) (cont'd)

Reconsidering its discrete counterpart

$$\sum_{n=1}^{N} \overleftarrow{a}_{mn}^{(r)} e^{-jk_r \Delta s(m-n)} \overleftarrow{u}_n^{(r)} = \overleftarrow{p}_m^{(r)} \quad \Rightarrow \quad \boxed{\overleftarrow{\mathbf{A}}^{(r)} \overleftarrow{\mathbf{u}}^{(r)} = \overleftarrow{\mathbf{p}}^{(r)}}$$

with $$\overleftarrow{a}_{hk}^{(r)} = a_{hk} \, e^{jk_r \Delta s(k-h)}$$

Δs sampling interval of the dummy varialble s

This expression permits to recover the envelope matrix $\overleftarrow{\mathbf{A}}$ ($\overleftarrow{\mathbf{M}}$, $\overleftarrow{\mathbf{K}}$) from the physical matrix **A** (**M**, **K**)

CEV EQUATION (summary)

From the discrete equation of a conservative dynamic problem

$$\mathbf{A}\,\mathbf{u} = \mathbf{p}$$

by:

load transformation: *i*) Fourier transform **F**, *ii*) application of bandpass windows $\mathbf{W}^{(r)}$, *iii*) wave number shift operation toward the origin $\mathbf{S}^{(r)}$, *iv*) inverse Fourier transform $\mathbf{F}^{-1}$:

$$\overleftarrow{\mathbf{p}}^{(r)} = \mathbf{F}^{-1}(\mathbf{S}^{(r)}\mathbf{W}^{(r)}\mathbf{F})\mathbf{p} \quad \Rightarrow \quad \overleftarrow{\mathbf{p}}^{(r)} = \mathbf{E}^{(r)}\mathbf{p}$$

matrix transformation $\overleftarrow{\mathbf{A}}^{(r)} = (\mathbf{S}^{(r)}\mathbf{A}\mathbf{S}^{*(r)})$

$$\Rightarrow \quad \boxed{\overleftarrow{\mathbf{A}}^{(r)}\overleftarrow{\mathbf{u}}^{(r)} = \overleftarrow{\mathbf{p}}^{(r)}}$$

CEV Equation

LIMITS OF CEV AND REMARKS

- The condition
  $\mathbf{a}(s,\sigma) = \mathbf{a}(s-\sigma)$
  is strictly valid for infinite systems.

- For finite systems it is not true.
  However it can be shown that it holds in an average sense, i.e. if the response is **spatially** averaged.

CEV EQUATIONS FOR NON CONSERVATIVE SYSTEMS

For non conservative systems (**A** and **u** are complex), the steps presented above can be maintained provided that **u**, **p** and **A** are written differently, as follows:

$$\overline{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_R \\ \mathbf{u}_I \end{bmatrix} \qquad \overline{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_R & -\mathbf{A}_I \\ \mathbf{A}_I & \mathbf{A}_R \end{bmatrix} \qquad \overleftarrow{\mathbf{p}} = \begin{bmatrix} \mathbf{p} \\ \mathbf{0} \end{bmatrix}$$

$$\boxed{\overline{\mathbf{A}}\,\overline{\mathbf{u}} = \overline{\mathbf{p}}} \quad \xrightarrow{\text{CEV}} \quad \overline{\overleftarrow{\mathbf{A}}}^{(r)}\,\overline{\overleftarrow{\mathbf{u}}}^{(r)} = \overline{\overleftarrow{\mathbf{p}}}^{(r)}$$

CEV

UNDERSAMPLIG

- The new problem, providing the complex envelope, has size *NxN*, identical to the original one – and has the same eigenvalues of the original one.
- Each spectral window has a low wavenumber spectrum, thus a fine mesh is unnecessary – and actually non convenient for the CEV application.

A suitable expansion matrix **R** is introduced to solve the problem at a low computational cost. By omitting the superscript *(r)* for the sake of simplicity:

$$\overleftarrow{\mathbf{u}} = \mathbf{R}\,\overleftarrow{\mathbf{u}}_{red}$$

R is rectangular and admits a pseudo-inverse **R**$^+$ such that

$$\mathbf{R}^+\mathbf{R} = \mathbf{I} \qquad \text{while} \qquad \mathbf{R}\mathbf{R}^+ \neq \mathbf{I}$$

The operation $\overleftarrow{\mathbf{A}}_{red} = \mathbf{R}^+\overleftarrow{\mathbf{A}}\mathbf{R}$ implies a partition of the original matrix into square submatrices, replacing each submatrix with a single value obtained by averaging its elements

For example, for a 6 x 6 matrix **A**, and a reduction ratio $\tau = 2$, we may choose

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so that, in this case it is $\mathbf{R}^+ = \mathbf{R}^T/\tau$

$$\overleftarrow{\mathbf{p}}_{red} = \mathbf{R}^+\overleftarrow{\mathbf{p}}$$

$$\overleftarrow{\mathbf{A}}_{red} = \mathbf{R}^+\overleftarrow{\mathbf{A}}\mathbf{R} \;; \qquad \overleftarrow{\mathbf{u}}_{red} = \mathbf{R}^+\overleftarrow{\mathbf{u}}$$

$$\overleftarrow{\mathbf{A}}_{red}\,\overleftarrow{\mathbf{u}}_{red} = \overleftarrow{\mathbf{p}}_{red}$$

RECOVERING THE PHYSICAL SOLUTION

The inverse relationship provides the physical solution

$$\overleftarrow{\mathbf{u}}^{(r)} = \mathbf{R}\,\overleftarrow{\mathbf{u}}^{(r)}_{red} \quad \Longrightarrow \quad \mathbf{u} = Re\left[\sum_r \mathbf{S}^* \overleftarrow{\mathbf{u}}^{(r)}\right]$$

Steps in CEV: summary

- **A** (**M** and **K**) is determined by a standard FEM

- The windowed envelop matrix is determined from $\overleftarrow{\mathbf{A}}^{(r)} = (\mathbf{S}^{(r)}\mathbf{A}\mathbf{S}^{*(r)})$
($\overleftarrow{\mathbf{A}}^{(r)}$ is not affected by the filtering operation)

- The windowed envelope load is obtained by FT of the forcing vector, windowing the wavenumber component by $W^{(r)}$, and shifting the spectrum by k_r. The IFT provides $\overleftarrow{\mathbf{p}}^{(r)}$

- $\overleftarrow{\mathbf{A}}^{(r)}_{red}$ is determined by $\overleftarrow{\mathbf{A}}^{(r)}_{red} = \mathbf{R}^{+}\overleftarrow{\mathbf{A}}\mathbf{R}$

- $\overleftarrow{\mathbf{p}}^{(r)}_{red}$ is determined by $\overleftarrow{\mathbf{p}}^{(r)}_{red} = \mathbf{R}^{+}\overleftarrow{\mathbf{p}}^{(r)}$

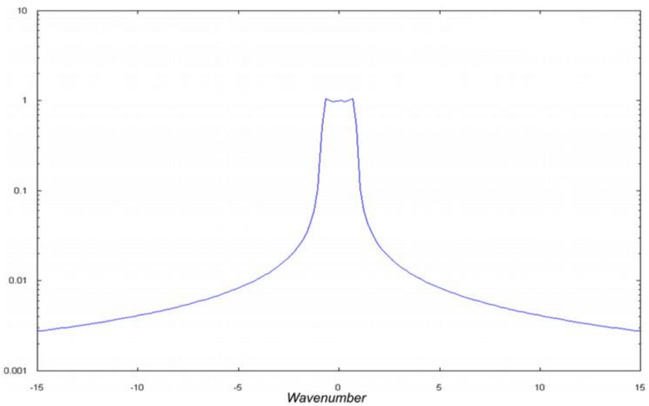
- From the CEV equation, $\overleftarrow{\mathbf{u}}^{(r)}$ is determined, and the physical response recovered

$$\mathbf{u} = Re\left[\sum_r \mathbf{S}^{*}\overleftarrow{\mathbf{u}}^{(r)}\right]$$

LIMITS OF CEV AND REMARKS

Looking at the spectrum of a point load (flat in the frequency domain), the filtered spectrum is not sharply rectangular, due to the use of a finite number of points, i.e. a truncated FT of the signal (leakage).

- Leakage effect of the filtering process, implying that CEV acts correctly on the forced part of the solution but does not tackle efficiently the homogeneous part

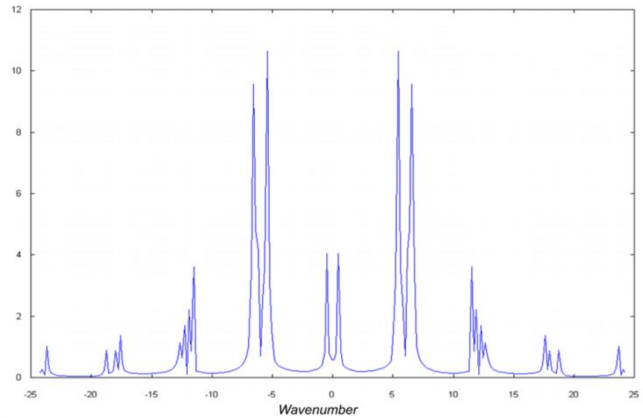


Filter spectrum shifted to the wavenumber origin

… the eigenvalues of $\overleftarrow{\mathbf{A}}$ are equal to those … the envelope solution matches perfectly

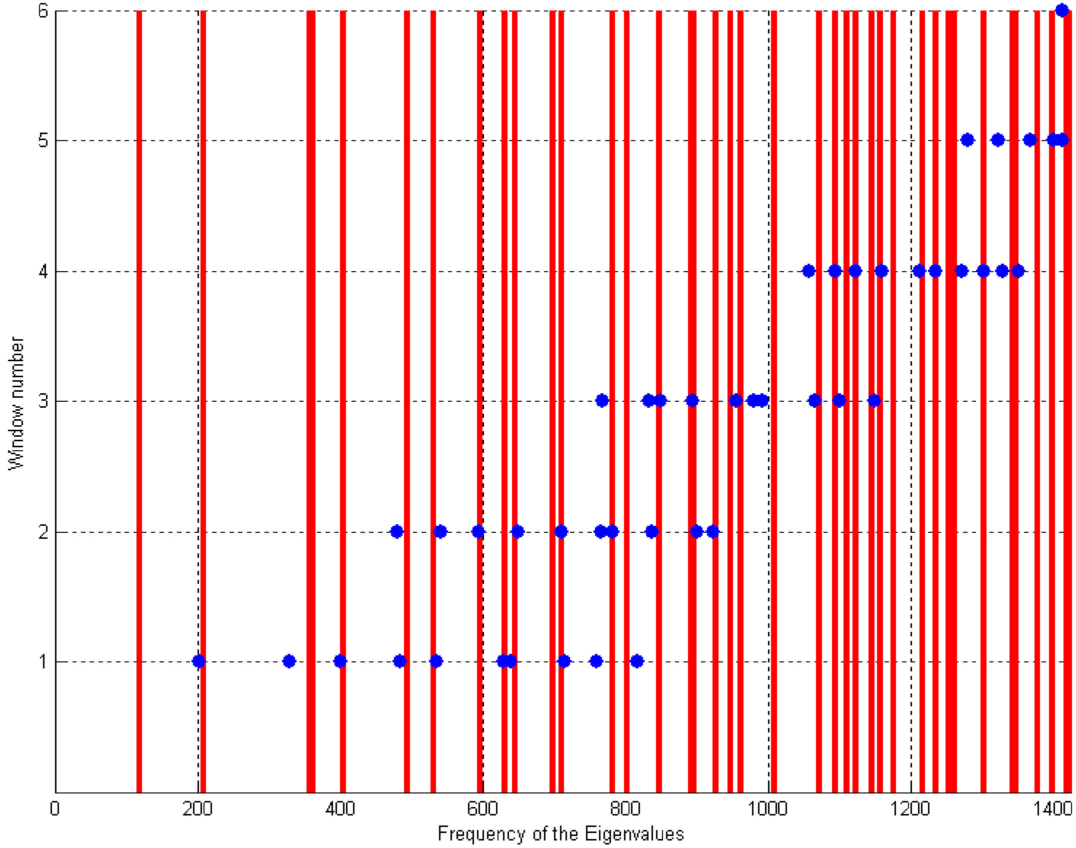
This is a drawback especially for low damping. However the contribution of different windows having different phases, reduces significantly such spread.

…educed, the eigenvalues of the reduced … from those of the physical problem: an …ing



Response of the filtered load (it is not strictly confined in the filter bandwidth)

…he reduced problem are a subset of the …d problem, under the assumption that the …g a set of windows in the wavenumber …oximate sufficiently well the eigenvalues …btaining a reasonable response.


6
5
4
3
2
1
Window number
0
200
400
600
800
1000
1200
1400
Frequency of the Eigenvalues

LIMITS OF CEV AND REMARKS

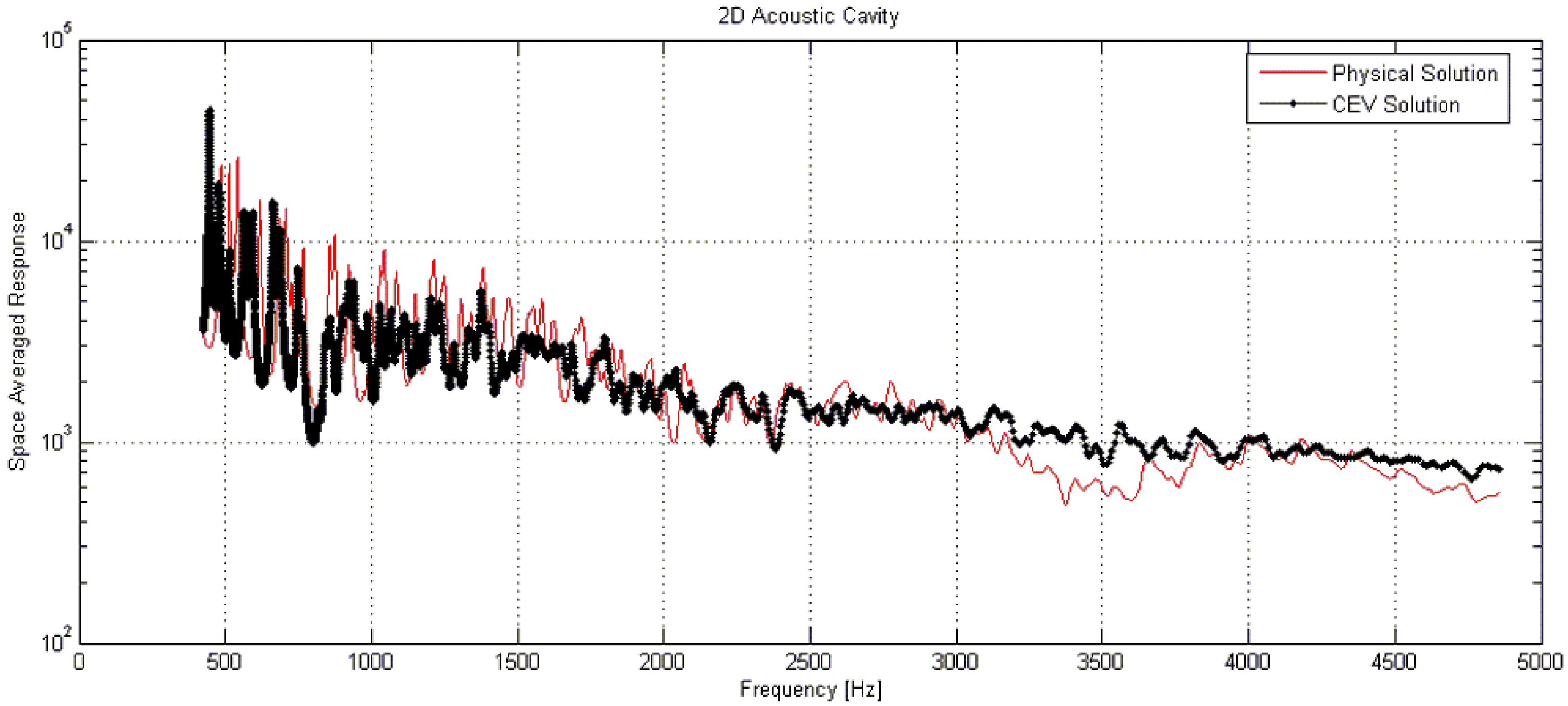
CEV can be considered successful when the modes of the system do not play a key role. Particularly:

- when the damping is relatively high
- when a direct field is preponderant with respect to the reverberant field
- for high frequency problems with an acceptable damping
- when an external acoustic problem (no modes) is considered*.

Application 1: 2D cavity with a point source

FE discretization: grid 64x64 (4096 DOF)
CEV discretization: grid 16x16 (512 DOF)

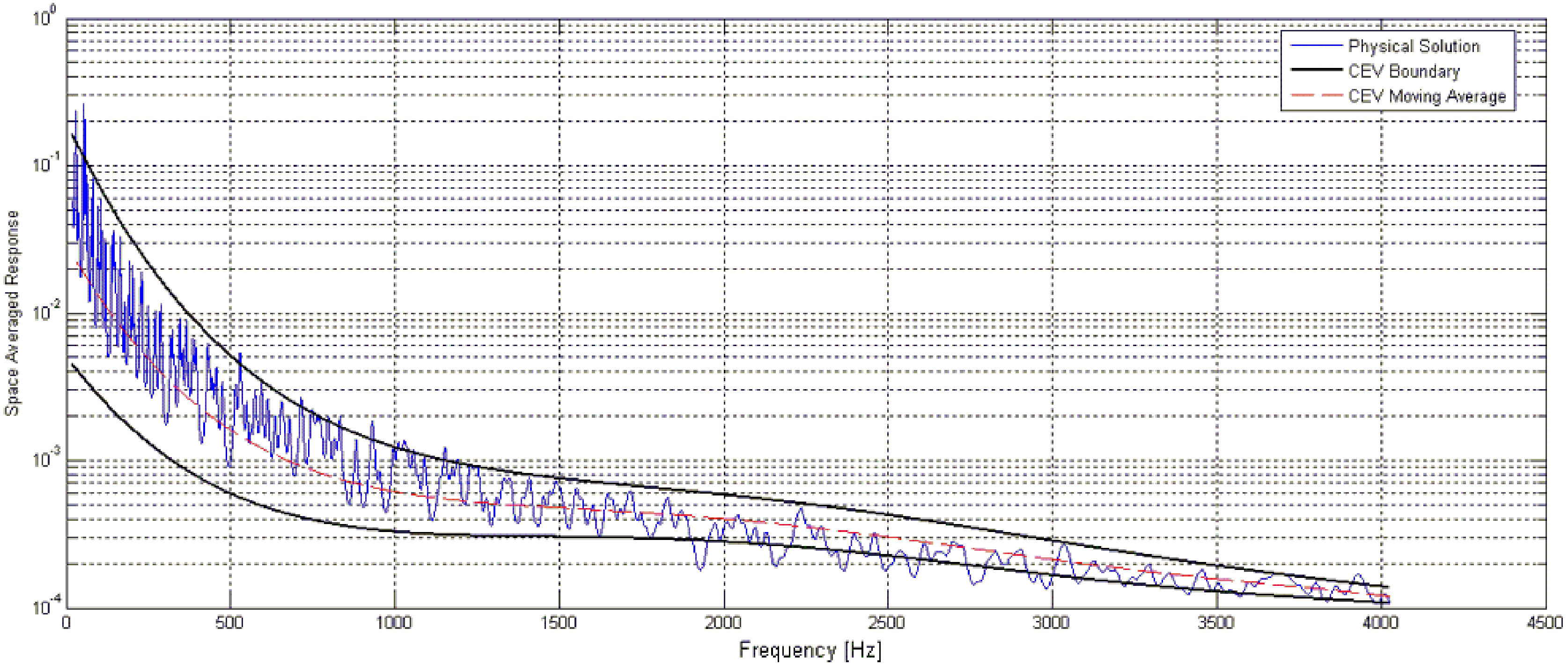
Damping 0.01



Application 2: bending plate 1x1x0.002 m with a point source

FE grid: 64x64 (12288 DOF). CEV DOFS: 384 (Reduction rate 32)

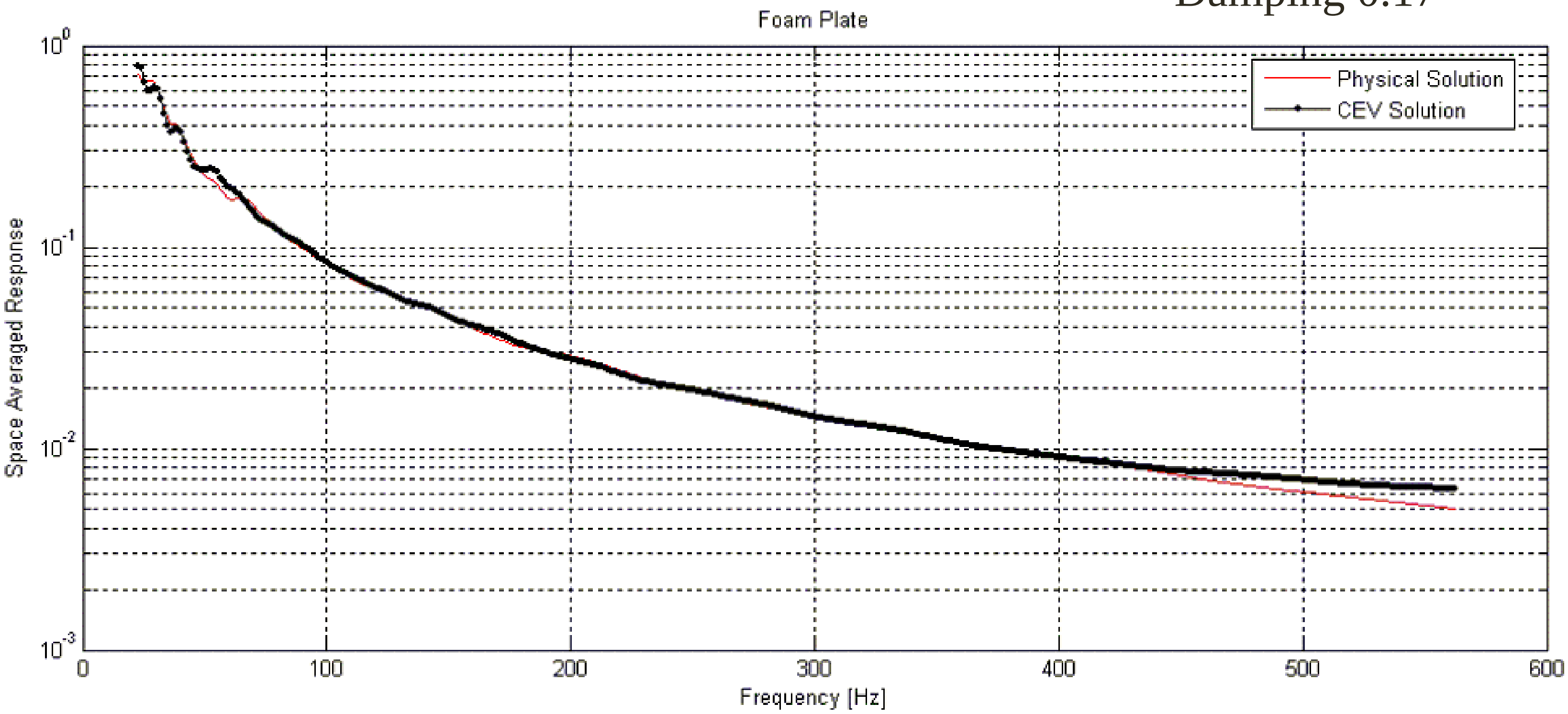
Damping 0.01



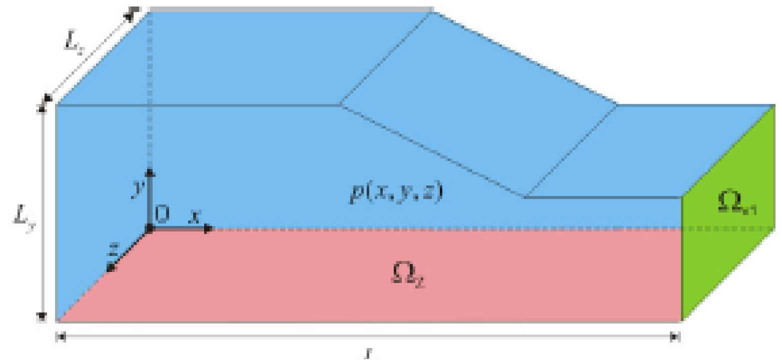
Application 3: foam plate 1x1x0.05 m with a point source

FE discretization: grid 64x64 (4096 DOF)
CEV discretization: grid 16x16 (512 DOF)

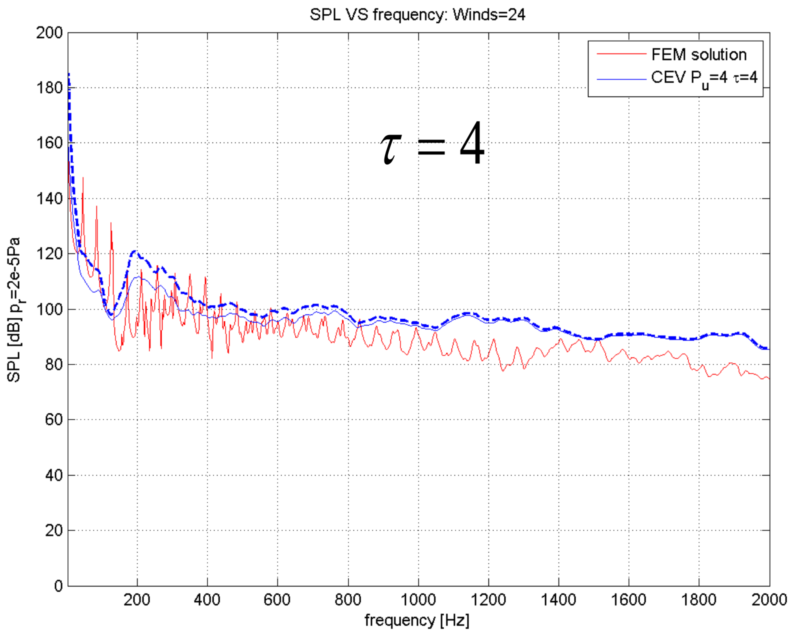
Damping 0.17

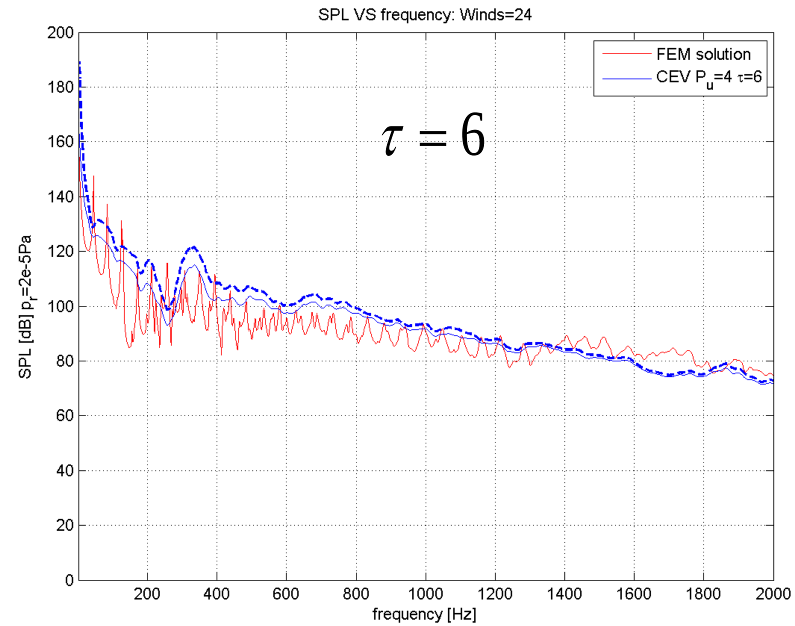


CAR CAVITY WITH IMPEDANCE ON THE FLOOR



A benchmark for the Marie Curie ITN

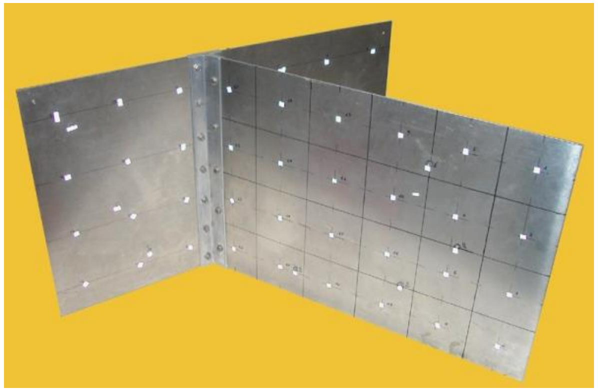


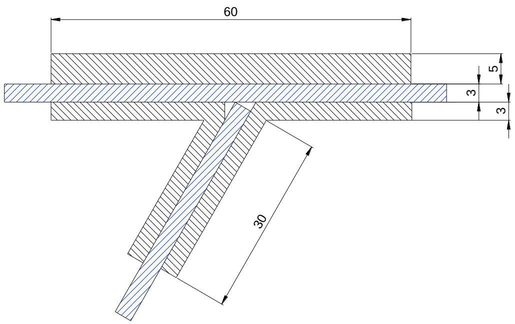


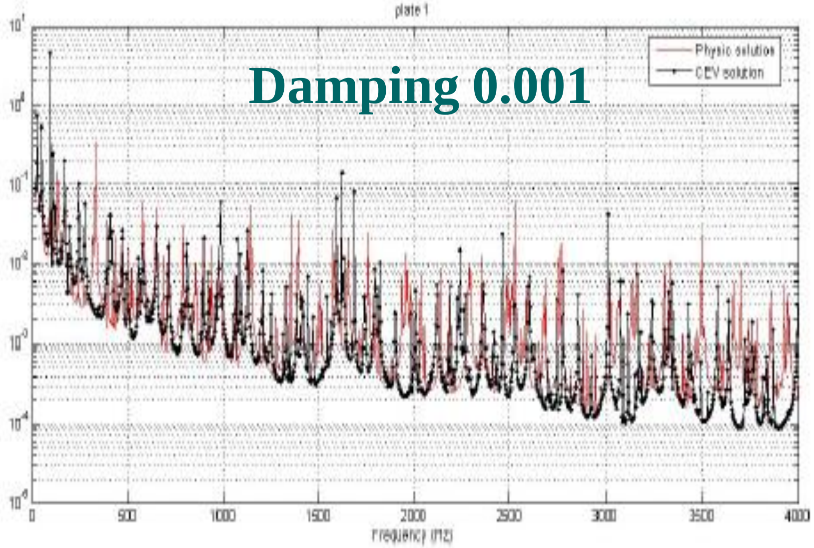
BENCHMARK FOR A PRIN NATIONAL PROJECT

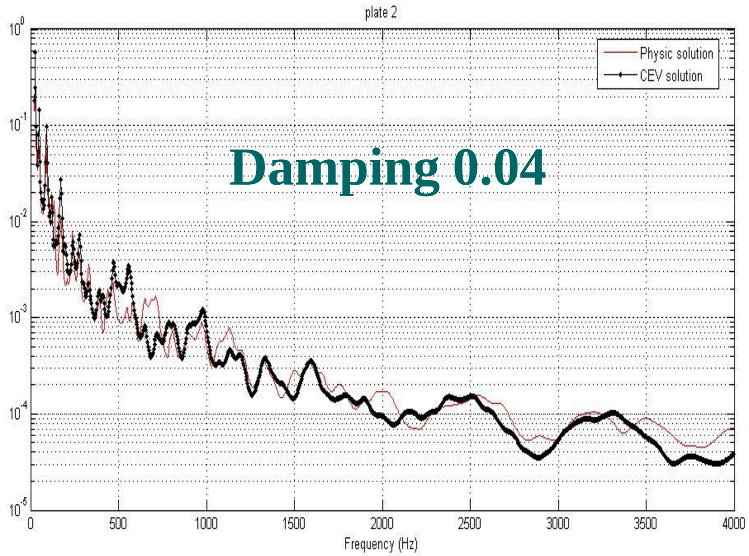
Reduction from 32226 dofs to 786 dofs (ratio 41)

Plate dimensions: 1- 600x400x3 mm, 2- 300x400x3, 3- 400x400x3.



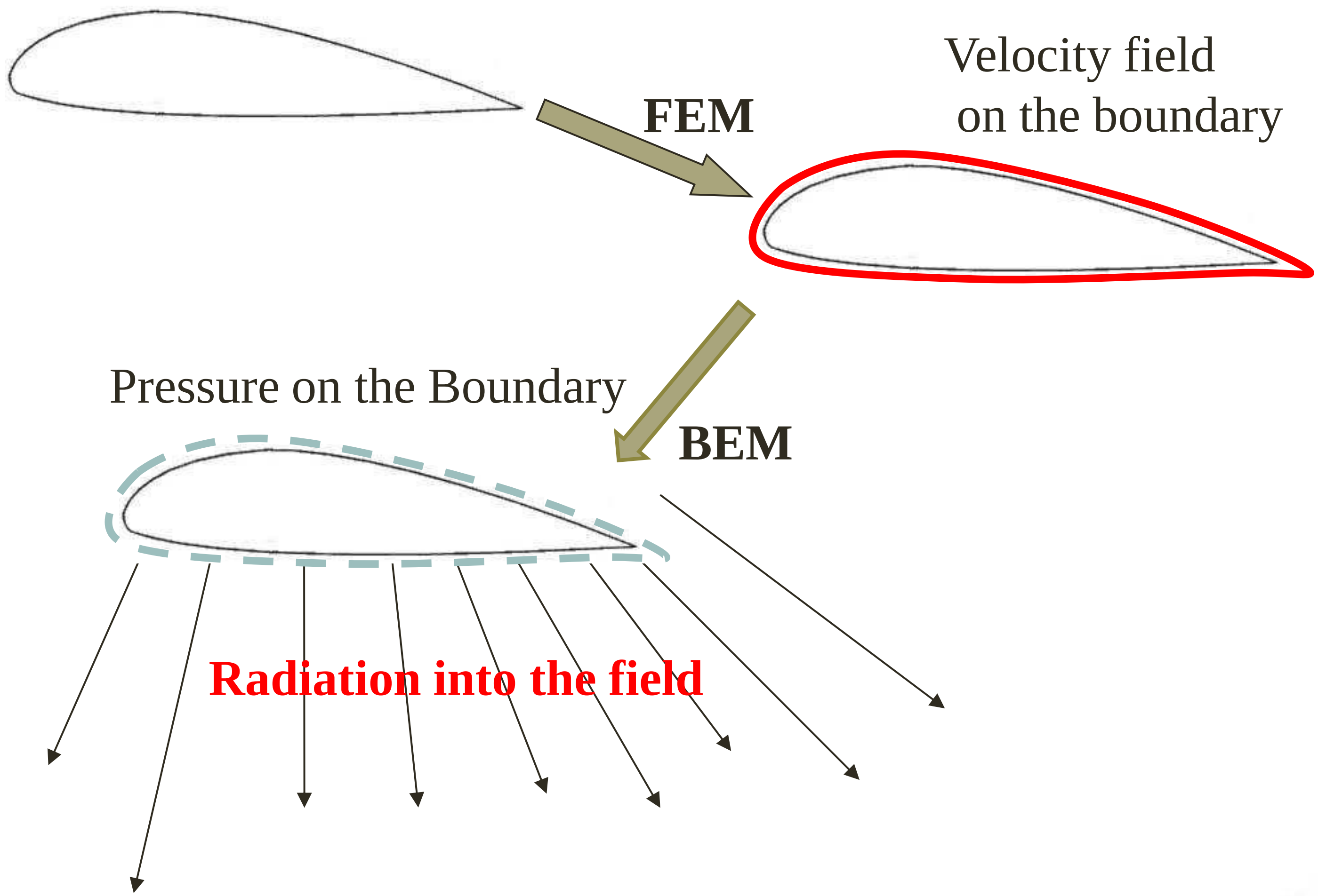






APPLICATION OF CEV TO BEM FORMULATION

Formulation of the external problem



APPLICATION OF CEV TO BEM FORMULATION

Integral formulation of a vibro-acoustic problem, under steady conditions, by the free-space Green function

$$\frac{1}{2}p(r_s) + \frac{1}{4\pi}\int_S p(r_s)\frac{\partial}{\partial n_s}\left(\frac{e^{-jkR}}{R}\right) = \int_S j\omega\rho_0 v_n(r_s)\frac{e^{-jkR}}{R}dS$$

Whatever the type of elements used, in matrix form one has

$$\mathbf{T}(\omega)\mathbf{p} = \mathbf{B}(\omega)\mathbf{v} = \mathbf{c}$$

This problem can be solved by following the same steps shown before, i.e.:

- a FEM is used to determine the response **v** (CEV can be used but not convenient)
- a BEM is used to determine the matrix **T** $\longrightarrow \overleftarrow{\mathbf{T}} \Rightarrow \overleftarrow{\mathbf{T}}_{red}$

  or
- a BEM is used to determine the matrix **T** $\longrightarrow \mathbf{T}_{red} \Rightarrow \overleftarrow{\mathbf{T}}_{red}$ more convenient computationally
- a BEM is used to compute **c** $\longrightarrow \overleftarrow{\mathbf{c}} \longrightarrow \overleftarrow{\mathbf{c}}_{red}$

APPLICATION OF CEV TO BEM FORMULATION

The CEV method applied to the boundary element formulation provides

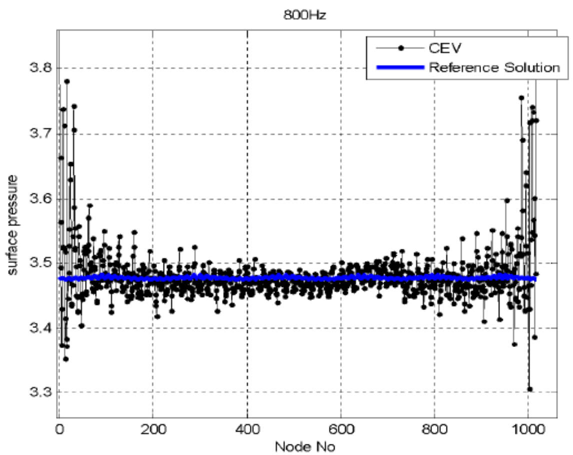
$$\overleftarrow{\mathbf{T}}_{red}\,\overleftarrow{\mathbf{p}}_{red} = \{\overleftarrow{\mathbf{B}}\,\overleftarrow{\mathbf{v}}\}_{red} = \overleftarrow{\mathbf{c}}_{red}$$

The mistuning does not affect the BEM-CEV formulation because the operator **T** is not directly related to the structural operator: <u>thus there is no error in the natural frequencies location</u>.

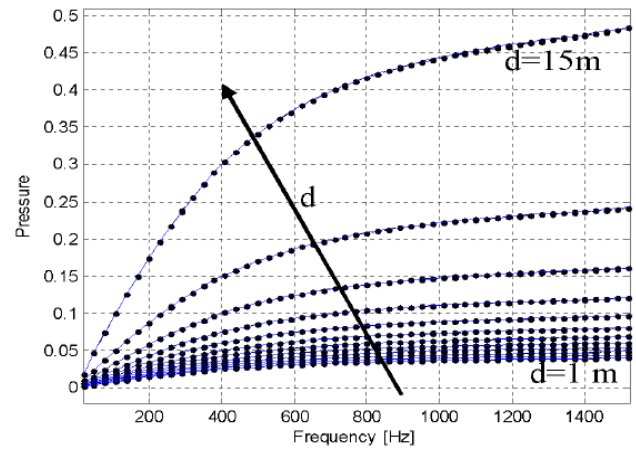
NO MISTUNING

TEST CASES: pulsating sphere (r = 0.1m, v_n= 0.01 m/s)

External field. Reduction factor 8: from 1016 to 127 DOFS

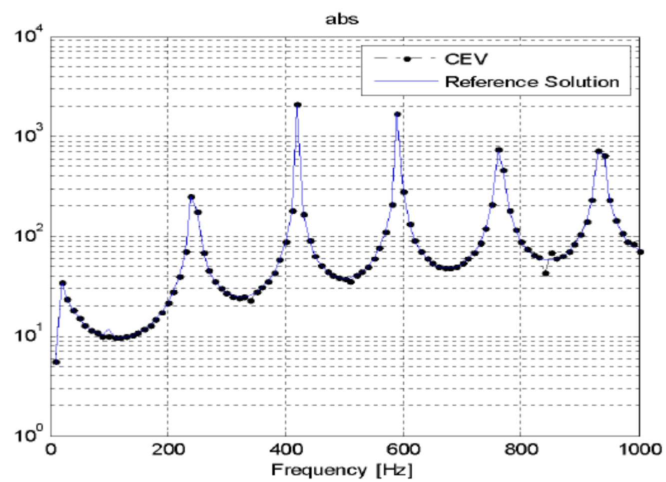


Pressure field on the surface
f = 800 Hz



External field

Internal field



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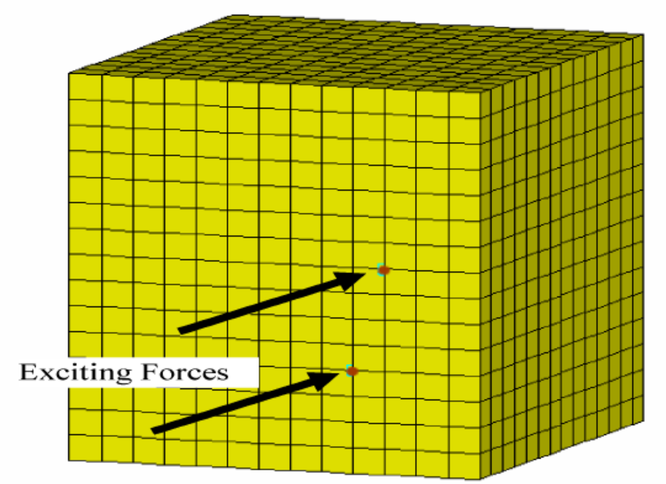
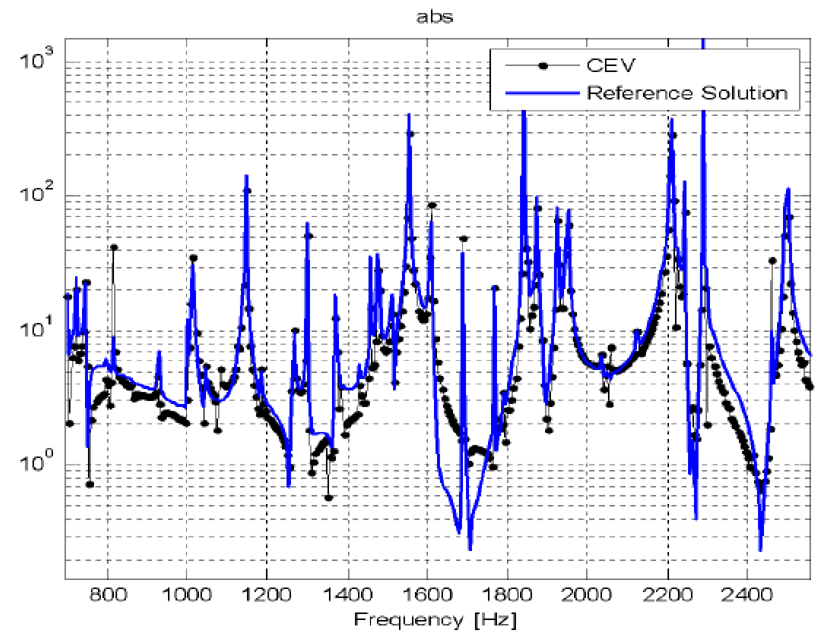
Lab. di Vibrazione e Acustica Strutturale

TEST CASES: external field generated by a loaded box

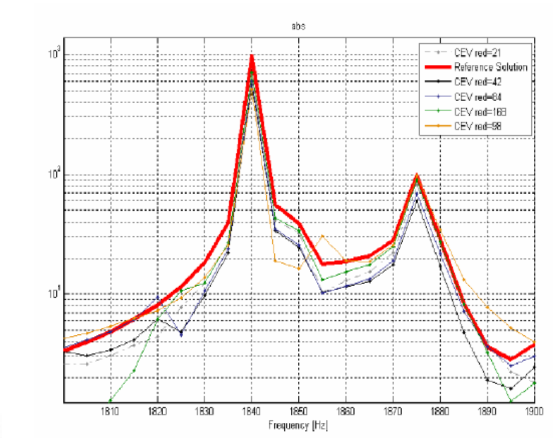


Force spectra are flat between 700 and 2500 Hz



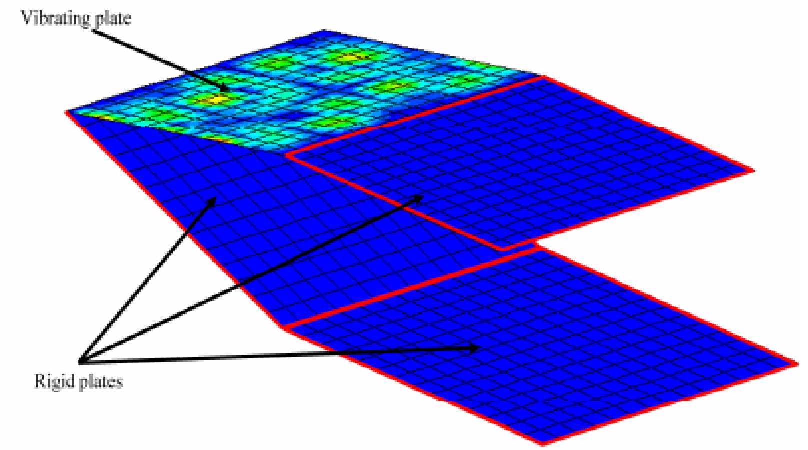
Pressure spectrum at a distance d=15m from the box

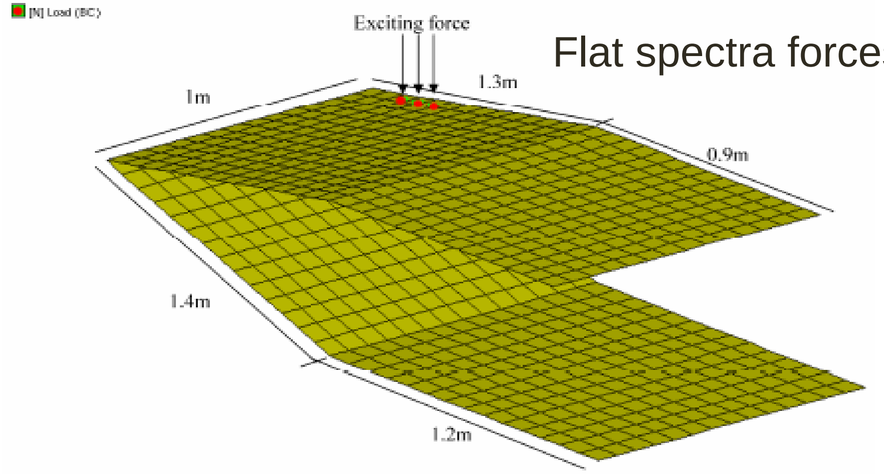
Reduction factor 21: from 1176 to 112 DOFS



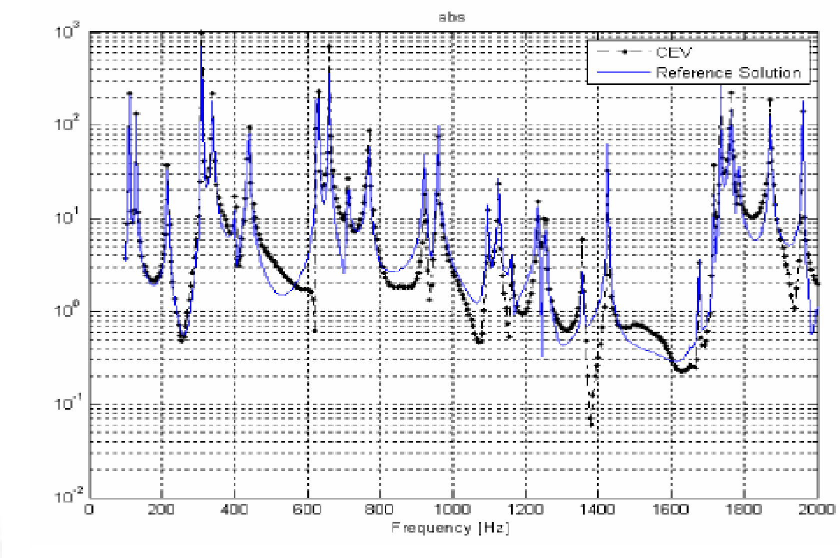
Point spectrum at 15 m from the box:
Comparison between reference solution and CEV for different reduction factors (21 – 168)
Note that for reduction factor 168, the CEV Dofs are only 14

TEST CASES: external field radiated by a benchmark





Flat spectra forces



Pressure spectrum at a reference point of the field

Reduction factor 19: from 855 to 45 DOFS

TEST CASES: external field radiated by a benchmark

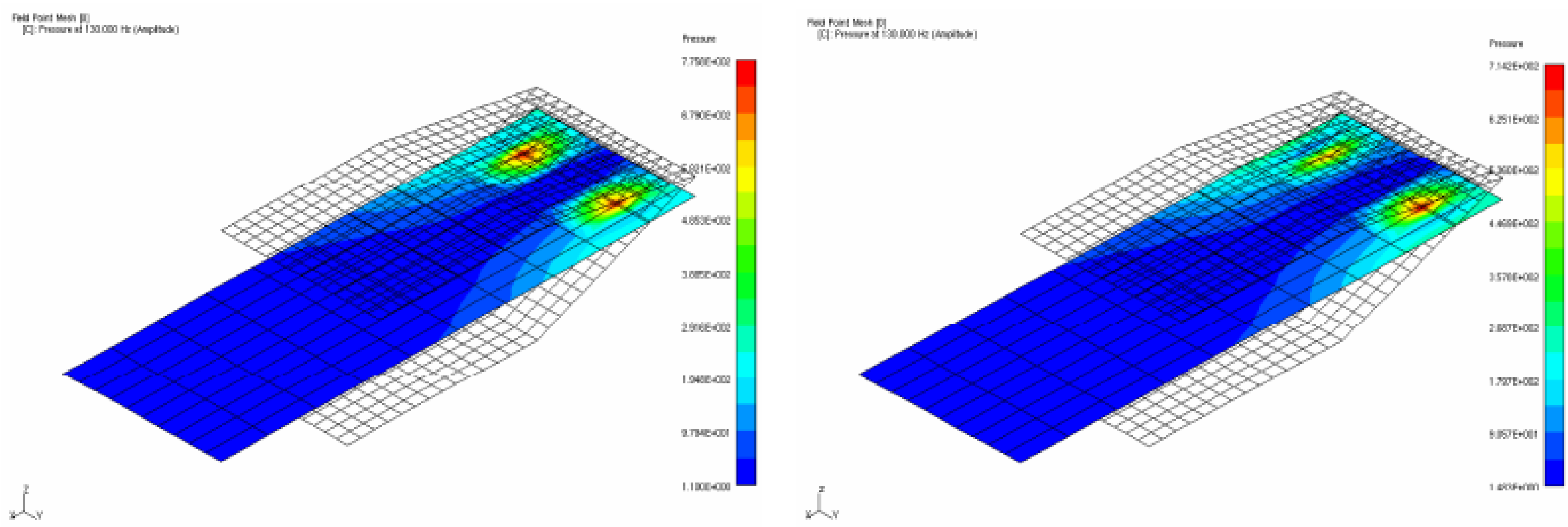


Figure 9: Internal pressure field at 130Hz. Left: reference solution, right: CEV solution

TEST CASES: external field radiated by a benchmark

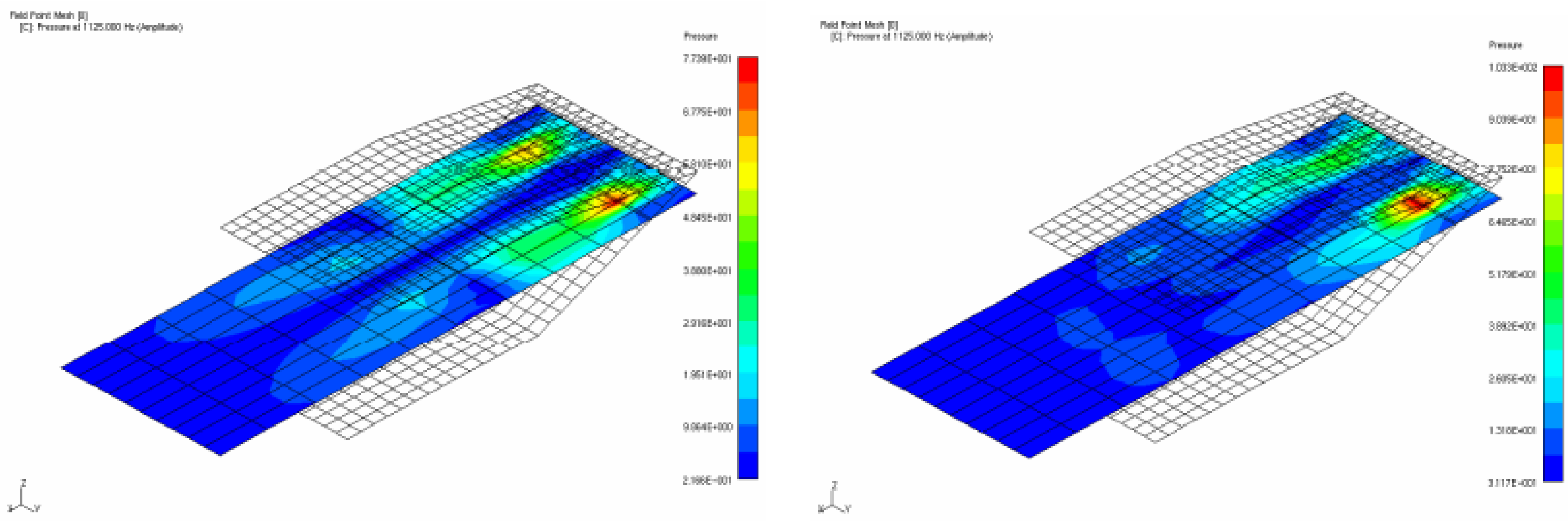


Figure 10: Internal pressure field at 1125Hz. Left: reference solution, right: CEV solution

A reference model for future developments (WKB*)

Possible connections with CEV, VTCR, DEA, WBM, WIA?

WKB (Eikonal) establishes a relationship between the exact equations of waves and the ray tracing approximation.

Considering the wave equation with a harmonic excitation, it is:

$$\nabla^2\psi + n^2k^2\psi = p$$

where *n(x)* is the refraction index: $c_0/c(\boldsymbol{x})$

By factorizing ψ into amplitude and phase $\psi(\mathbf{x}) = A(\mathbf{x})\, e^{-jS(\boldsymbol{x})}$,

substituting into the wave equation and getting real and imaginary parts:

$$\begin{cases} \nabla^2 A - k^2 A\left(|\nabla S|^2 - n^2\right) = 0 \\ 2\left(\nabla A \cdot \nabla S\right) + A\nabla^2 S = 0 \end{cases}$$

(* Wenzel, Kramers and Brillouin)

WKB (cont.)

Let $\varepsilon = 1/k^2$:

$$\begin{cases} \epsilon \nabla^2 A - A\left(|\nabla S|^2 - n^2\right) = 0 \\ 2\left(\nabla A \cdot \nabla S\right) + A \nabla^2 S = 0 \end{cases}$$

By expressing the solution in the form

$$A = A_0 + \frac{1}{k^2} A_1, \qquad S = S_0 + \frac{1}{k^2} S_1$$

and using a perturbation approach up the first order:

$$\text{at the zero-th order} \begin{cases} |\nabla S_0(\mathbf{x})|^2 - n^2(\mathbf{x}) = 0 \qquad \rightarrow \qquad |\nabla S_0(\mathbf{x})| = n(\mathbf{x}) \\ 2n \nabla A_0 \mathbf{r} + A_0 \nabla \cdot (n\mathbf{r}) = 0 \end{cases}$$

$$\text{at the first order} \begin{cases} \nabla^2 A_0 - 2 A_0 n \nabla S_1 \mathbf{r} = 0 \\ 2\left(\nabla A_0 \cdot \nabla S_1 + n \nabla A_1 \cdot \mathbf{r}\right) + A_0 \nabla^2 S_1 + A_1 \nabla \cdot (n\mathbf{r}) = 0 \end{cases}$$

WKB (cont.)

At order zero, i.e. $\lambda \rightarrow 0$, i.e. $k \rightarrow \infty$ (ray tracing)

$$|\nabla S(\mathbf{x})|^2 - n^2(\mathbf{x}) = 0 \qquad \rightarrow \qquad |\nabla S(\mathbf{x})| = n(\mathbf{x})$$

Equal phase surfaces are those over which *S(**x**)* are constant: rays are those lines intersecting orthogonally equal phase surfaces. The unit vector along the ray is:

$$\mathbf{r} = \frac{\nabla S_0}{|\nabla S_0|} \qquad \rightarrow \qquad \nabla S_0 = n\mathbf{r}$$

producing:

$$\begin{cases} \nabla S = n\mathbf{r} \\ 2\left(\nabla A \cdot n\mathbf{r}\right) + A\nabla\left(n\mathbf{r}\right) = 0 \end{cases}$$

that can be solved iteratively.

WKB (cont.)

Turning this approach to discrete form:

$$\nabla^2\psi + n^2k^2\psi = p \qquad \rightarrow \qquad \mathbf{L}\psi = \mathrm{p}$$

$$\psi(\mathbf{x}) = A(\mathbf{x})e^{jS(\mathbf{x})} \qquad \rightarrow \qquad \psi = \mathbf{SA}$$

with: $\mathrm{S} = \begin{bmatrix} e^{jS(\mathbf{x}_1)} & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & e^{jS(\mathbf{x}_p)} \end{bmatrix}$ and $\mathrm{a} = \begin{bmatrix} A(\mathbf{x}_1) \\ \cdots \\ A(\mathbf{x}_p) \end{bmatrix}$

i.e. **(L S) a = p**

LS acts on the wave operator by modulating exponentially its coefficients, just as CEV and others methods do.

Comparison between WKB and CEV

WKB

- It uses different orders of perturbation to approximate the solution

$$A = A_0 + \frac{1}{k^2} A_1, \qquad S = S_0 + \frac{1}{k^2} S_1$$

- *S*(***x***) is unknown

CEV

- It uses several windows centered on k_r to shift the HF components and approximate the solution

$$\overleftarrow{u}^{(r)}(s) = F^{-1}\{\overleftarrow{U}^{(r)}(k)\} = F^{-1}\{U(k+k_r)W^{(r)}(k+k_r)\}$$

$$u(s) = 2\,\mathrm{Re}\{\sum_{r=1}^{P} \overleftarrow{u}^{(r)}(s)\; e^{j k_r s}\}$$

- The phase modulating term is assigned

Comparison between WKB and VTCR/WBM*

WKB	VTCR/WBM
In WKB the solution is represented by waves amplitudes modulated by phase functions, and different perturbation orders are used to approximate the solution.	Both VTCR and WBM use shape functions that are exact solution of the governing differential equations - modes represented by a superposition of wave shape functions in WBM, local modes, superposition of appropriate waves in VTCR. In VTCR a two-scale approximation is used in the weak formulation. Only the slow scale (wave amplitude) is discretized while the fast varying scale (spatial shape of the wave) is analytically described.

* WBM: Wave based method, VTCR: Variational theory of complex rays

Comparison between WKB and DEA/WIA*

WKB	DEA/WIA
In WKB the zero-th order approximation provides the ray tracing technique. Using higher orders of perturbation it is possible to refine the solution.	In DEA the ray dynamics is described by a set of basis functions. When using the lowest basis function the ray tracing approximation, and SEA, is obtained In WIA the displacement field is obtained by a superposition of travelling waves. By neglecting phase dependencies, only an energy beam is associated to each wave. By expanding each beam by a Fourier series, an energy balance equations is obtained. It provides the SEA equations if the series is arrested to the first term.

*DEA: Dynamic Energy analysis, WIA: Wave intensity approach

CONCLUDING REMARKS (1)

- CEV does not use an energy formulation, but performs a transformation leading to a new variable that has a low wavenumber content and capable of accounting correctly for the boundary conditions. As such, it is particularly appropriate to deal with high frequency problems.

- The envelope mass and stiffness matrices are determined directly from the FE matrices, so that any commercial FE or BE code can be used.

- A reduction technique is applied to these matrices to get a new problem whose dimensions are much smaller than the original one, with a "relevant" saving of time computation.

- The approach is particularly convenient when the modes do not play a relevant role: e.g. external acoustic problems

CONCLUDING REMARKS (2)

- CEV can be applied to both external and internal vibro-acoustic problems, with errors bounded within 3 dB from the reference solution.

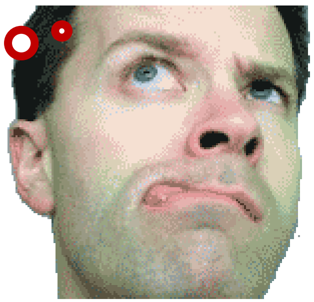
- Unlike the application of CEV with FEM, in the application to BEM the mistuning problem is absent. Thus, CEV can be efficiently used to determine the internal acoustic field.

- The reduction factor can be increased at will, without affecting significantly the quality of the solution: however, increasing the reduction factor it is necessary to increase the number of spectral filters.

- Shall we define a common goal and tasks to address predictive methods, with clear tasks and goals?

Time average?
Frequency average?
Ensemble average?

Direct field?
Reverberant field?
Wave interference?



Thank you for your attention

Any question?

Weak form of the kernel

For an infinite system *a(s,σ) = a(s-σ)* (freee space Green function)
For finite systems it does generally hold.
However it can hold in a weak or average form, i.e.

$$\langle a(s,\sigma)\rangle = \langle \tilde{a}(s-\sigma)\rangle$$

By FT *a(s,σ)* ⟶ $\int_{-\infty}^{\infty} a\,(s,\sigma)\, e^{-ik_x x} ds = A(k_x,\sigma)$

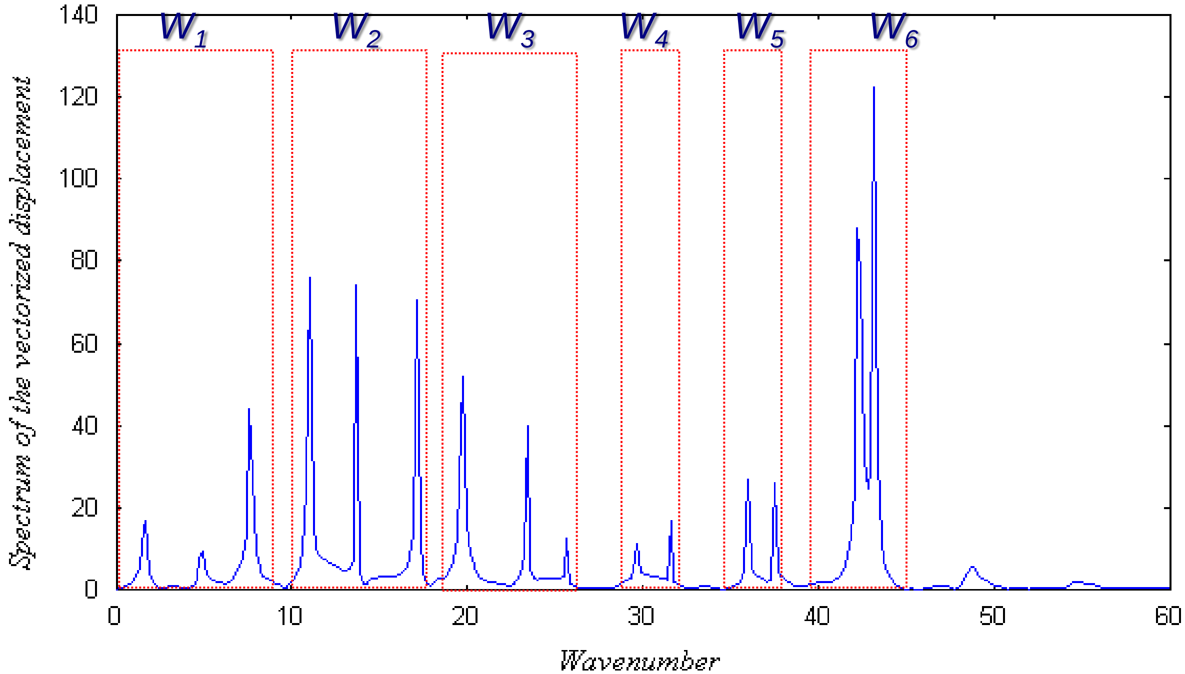
By FT *ã(s-σ)* ⟶ $\int_{-\infty}^{\infty} \tilde{a}\,(s-\sigma)\, e^{-ik_x x} ds = \tilde{A}(k_x)\, e^{-ik_x\sigma}$

i.e., to satisfy the convolution condition it must be $A(k_x,\sigma)=A(k_x)\, e^{-ik_x\sigma}$

⟹ $\tilde{A}(k_x) = \left\langle \frac{A(k_x,\sigma)}{\langle e^{-ik_x\sigma}\rangle} \right\rangle$

By IFT, it is possible to determine *ã(s)* as: $\tilde{a}(s) = F^{-1}\left\{\frac{\langle A(k_x,\sigma)\rangle}{\langle e^{-ik_x\sigma}\rangle}\right\}$

SPECTRAL WINDOW DECOMPOSITION



W_1
LOW FREQUENCY SOLUTION N. 1

…..

W_6
LOW FREQUENCY SOLUTION N. 6

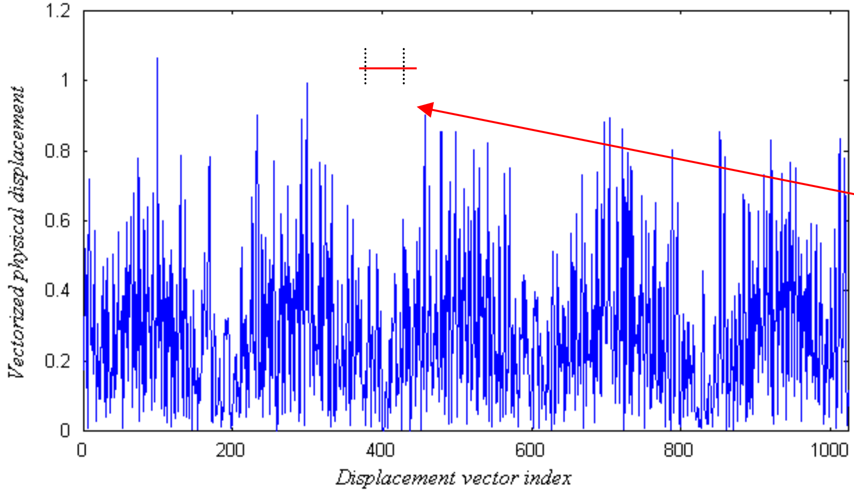
FORWARD SHIFTING

MOVE THE SOLUTIONS AT THE RIGTH WN-

+

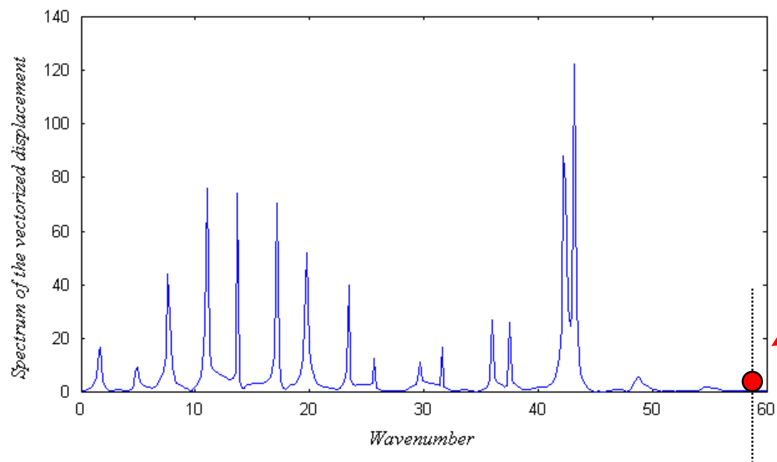
HIGH FREQUENCY PROBLEMS & SPECTRAL CHARACTERISTICS

FOURIER TRANSFORM



δs

space step

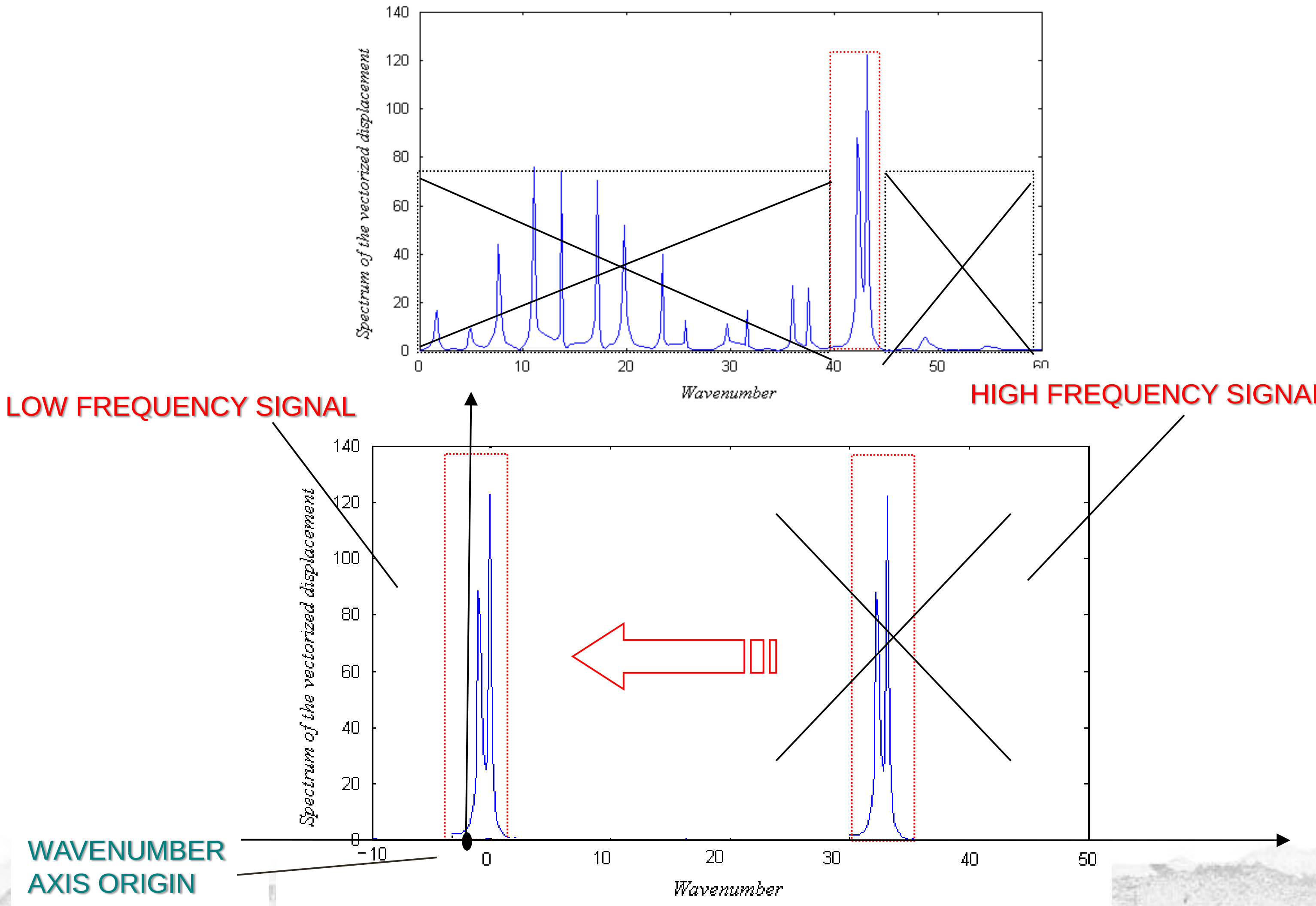


k_{MAX}

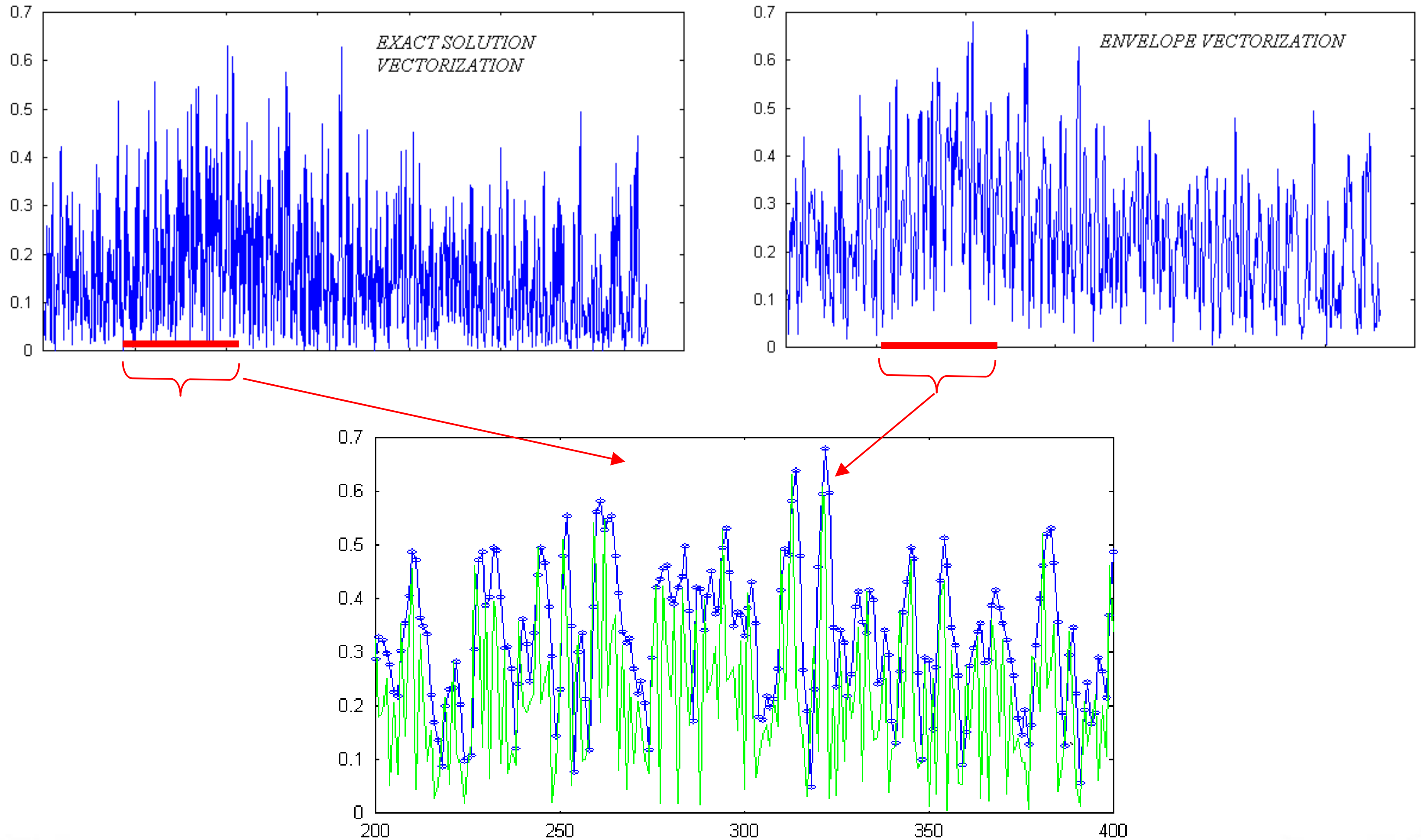
maximum wavenumber

CEDV SOLUTION:

- **DESCRIBE THE RESPONSE BY A SET OF NARROW BAND SIGNALS**

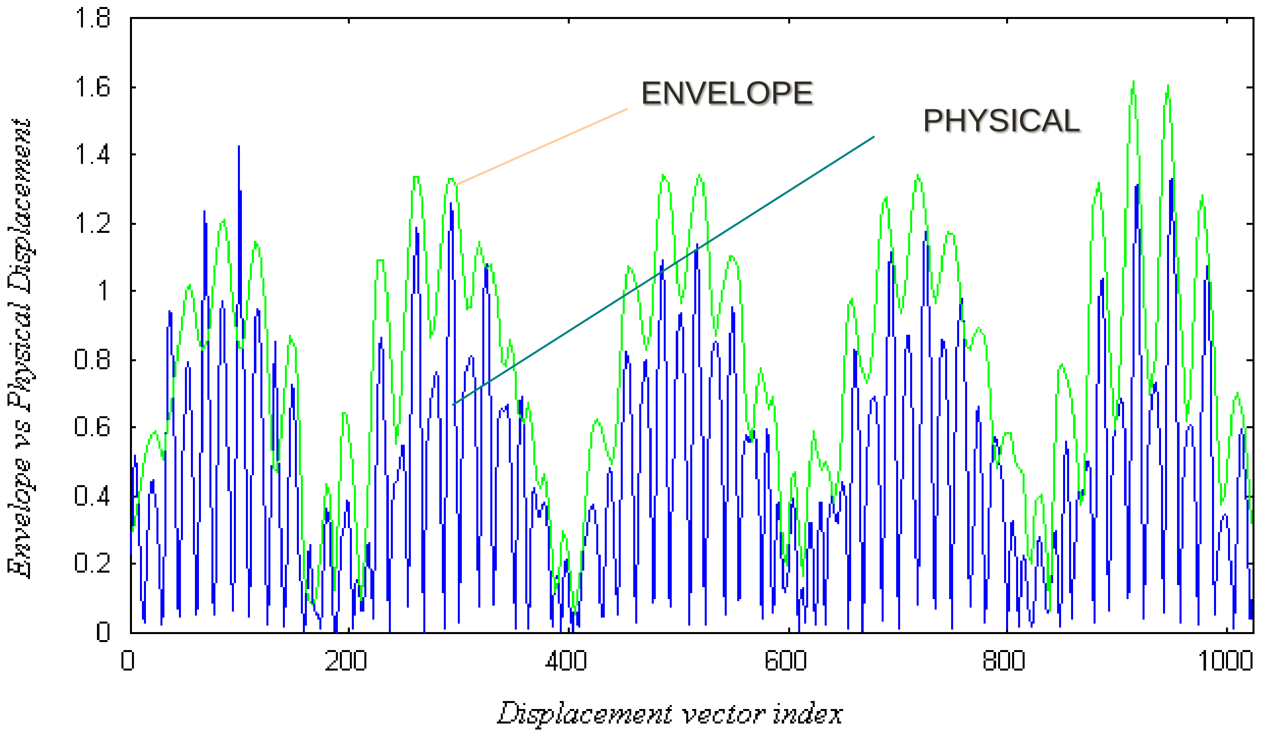


NUMERICAL RESULTS (34x34)



NUMERICAL RESULTS, PLATE (34x34 vs 16x16)

COMPARISON BETWEEN VECTORIZED SOLUTIONS



CEV: GOVERNING EQUATIONS

$$\tilde{x}(s) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x(s-\xi)}{\xi}d\xi \qquad \hat{x}(s) = x(s) + j\tilde{x}(s) \qquad \overleftarrow{x}(s) = \hat{x}(s)e^{-jk_0 x}$$

New equation of motion

$$\int_I a(s,\sigma)e^{jk_0(s-\sigma)}\overleftarrow{x}(\sigma)d\sigma = \overleftarrow{f}(s)$$

Equation of motion for the discrete counterpart

$$(-\omega^2\overleftarrow{\boldsymbol{M}} + \overleftarrow{\boldsymbol{K}})\overleftarrow{\boldsymbol{x}} = \overleftarrow{\boldsymbol{f}}$$

$$\overleftarrow{M}_{ij} = M_{ij}e^{jk_0\Delta l(j-i)} \qquad \overleftarrow{K}_{ij} = K_{ij}e^{jk_0\Delta l(j-i)}$$

CEV: ENVELOPE VECTORIZATION

$$(-\omega^2\mathbf{M}+\mathbf{K})\mathbf{w}=\mathbf{f} \quad \Longrightarrow \quad (-\omega^2\overleftarrow{\mathbf{M}}+\overleftarrow{\mathbf{K}})\overleftarrow{\mathbf{w}}=\overleftarrow{\mathbf{f}}$$

$$\overleftarrow{w}=\boldsymbol{E}[w]=(w+j\tilde{w})e^{-jk_0x}$$

$$\overleftarrow{M}_{ij}=M_{ij}e^{jk_0\Delta l(j-i)}$$

$$\overleftarrow{K}_{ij}=K_{ij}e^{jk_0\Delta l(j-i)}$$

Spectrum of the vectorized displacement of high dimensional problems