Trefftz and Weak Trefftz methods for the resolution of medium frequency problems

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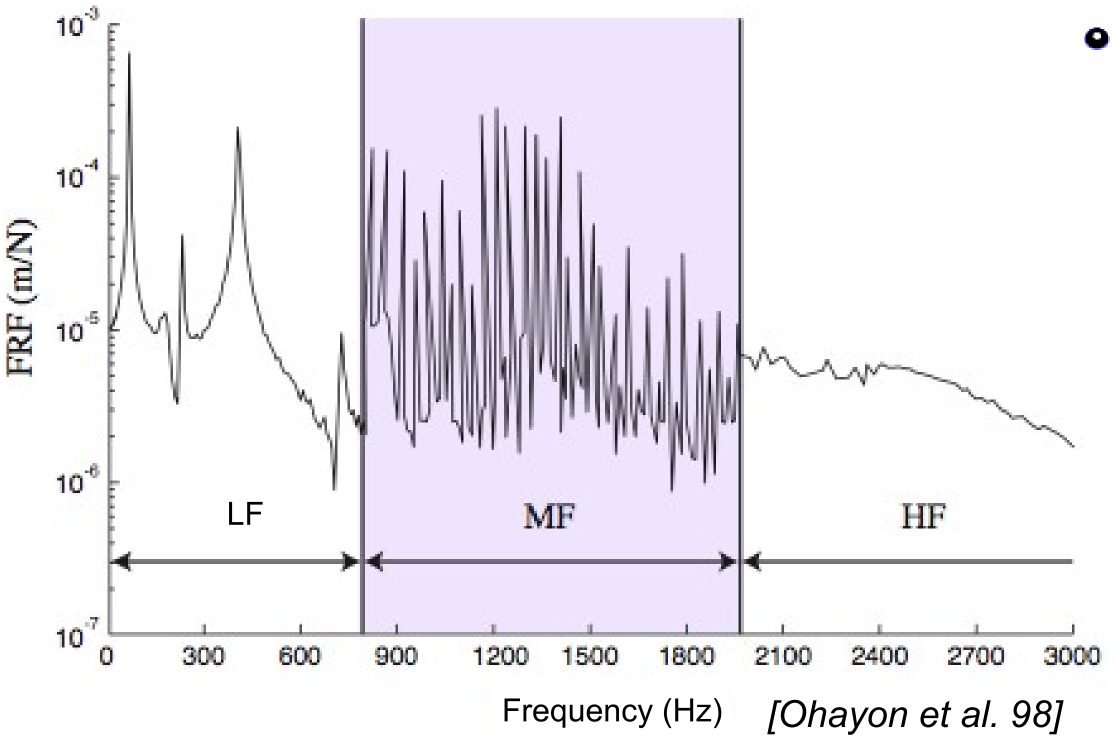
Medium frequency problems



[Ohayon et al. 98]

- Medium frequencies :
  - Many dozen of wavelengths in the structure
  - Modal overlap beginning
  - Boundary condition and structural parameter sensibility
  - Need for local quantities

Treffz ?

Erich Trefftz
Born 1888, Leipzig (Germany)

1906 : technical university of Aachen (mechanical engineering)
Became an eminent mechanician and applied mathematician
(few in second half of 19^{th} century)
Belong to the birth of numerical mathematics
(Ritz, Galerkin, Runge, …)
Worked in universities in Gottingen, New York, Strasbourg, with von Mises and Pandtl
1913: PhD, 1919: full professor Aachen
1922: full professor with chair in technical university of Dresden
1929: became an honourable doctor in Stuttgart
Died in 1937

“A mathematical problem can only be said to be solved totally if - at the end - results can be produced in the form of numbers”

Treffz ?

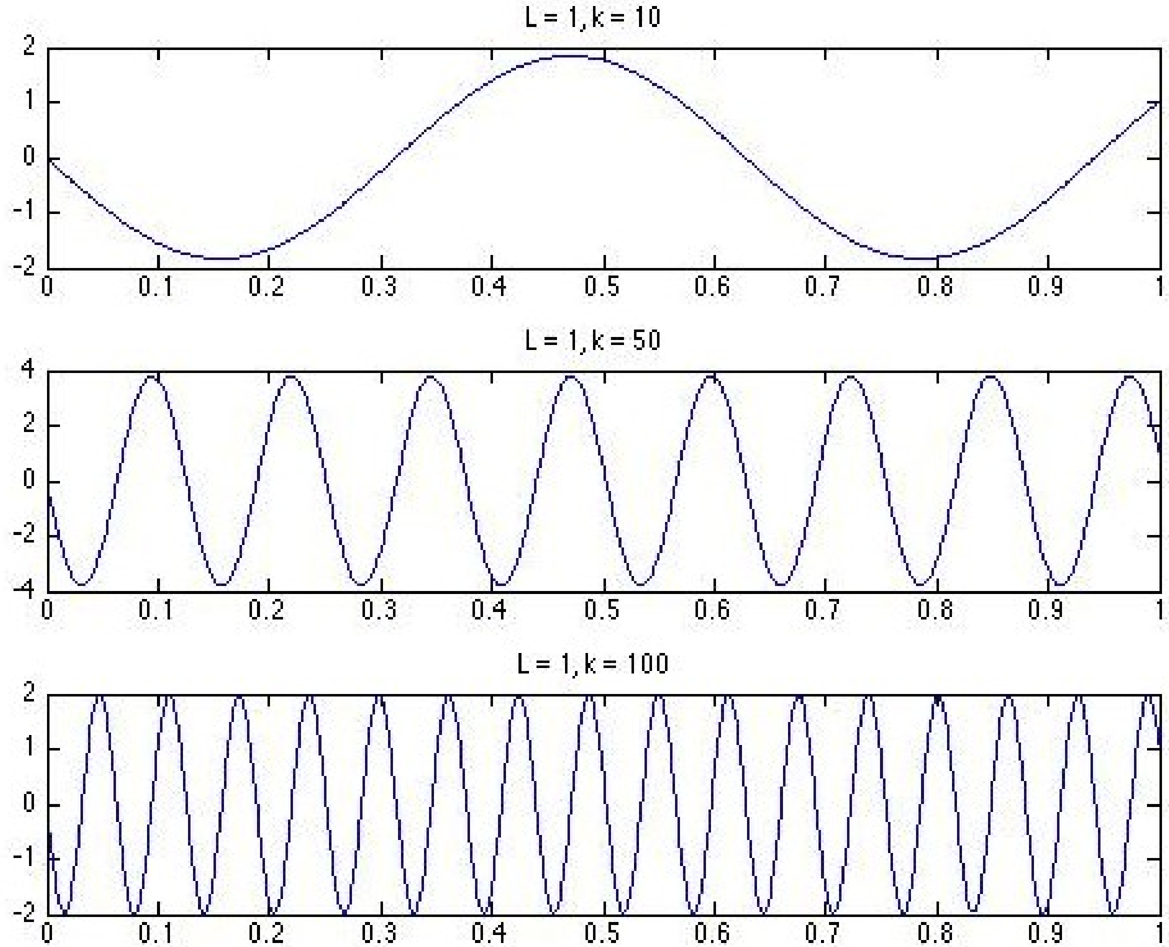
Research fields:
Hydrodynamics, applied mathematics, vibration theory, elasticity, buckling, variational methods, test functions, …

Mathematical works: always driven by technical applications:
1915: improvement of the Picard method as a successive solution approximation of ordinary differential equations
1919: solution of the potential and bipotential equations in a nearly circular domain with given boundary data
1926: **"Ein Gegenstück zum Ritzschen Verfahren"** lectured in Zürich on the 2nd International Congress of Technical Mechanics (later called the ICTAM — congress of IUTAM)

Saint Venant problem of torsion: find an approximated solution with test function which satisfy the governing equation, and not the geometrical boundary conditions.

Traditional resolution of DPE

$$\begin{cases} \dfrac{d^2u}{dx^2} + k^2 u = 0 \text{ in } [0;L] \\ u(x=0) = 0 \\ u(x=L) = u_L \end{cases}$$



$$u_{ex} = A e^{ikx} + B e^{-ikx}$$

Boundary conditions =>

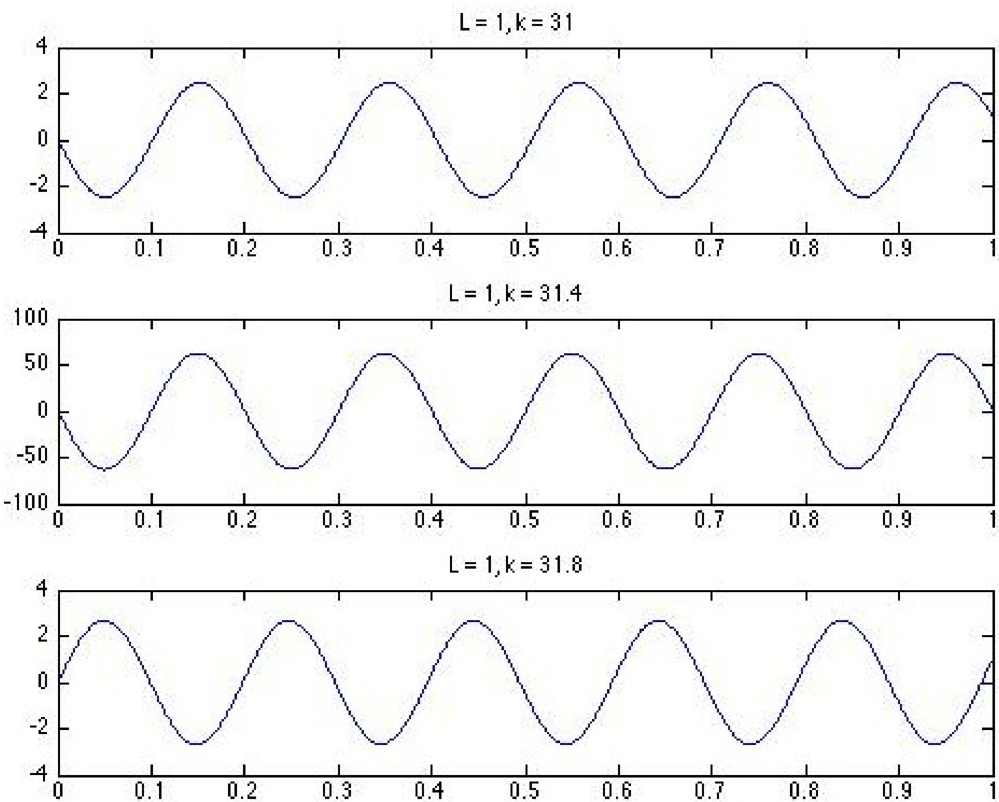
$$\begin{cases} A + B = 0 \\ A e^{ikL} + B e^{-ikL} = u_L \end{cases}$$

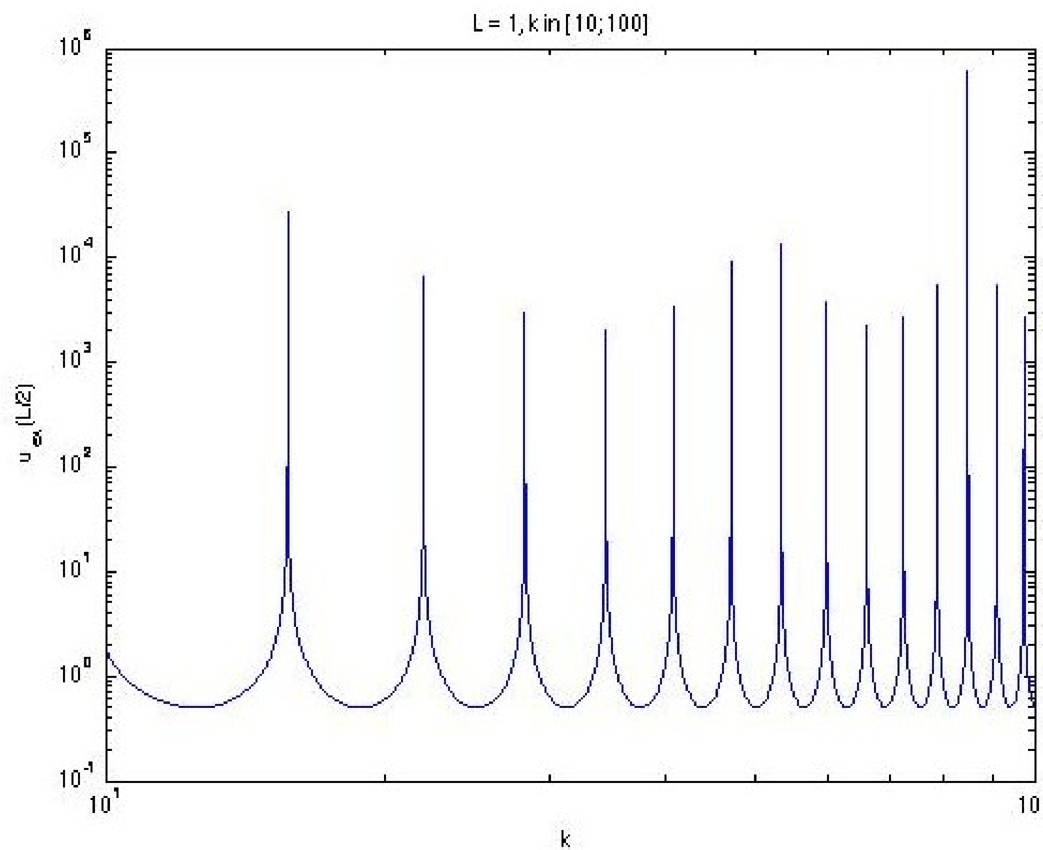
$$u_{ex} = \frac{e^{ikx} - e^{-ikx}}{e^{ikL} - e^{-ikL}}$$

Traditional resolution of DPE

$$u_{ex} = \frac{e^{ikx} - e^{-ikx}}{e^{ikL} - e^{-ikL}} \Rightarrow u_{ex} = \sin(kx)/\sin(kL)$$

Resonance if $kL = n\pi \quad n \in '$

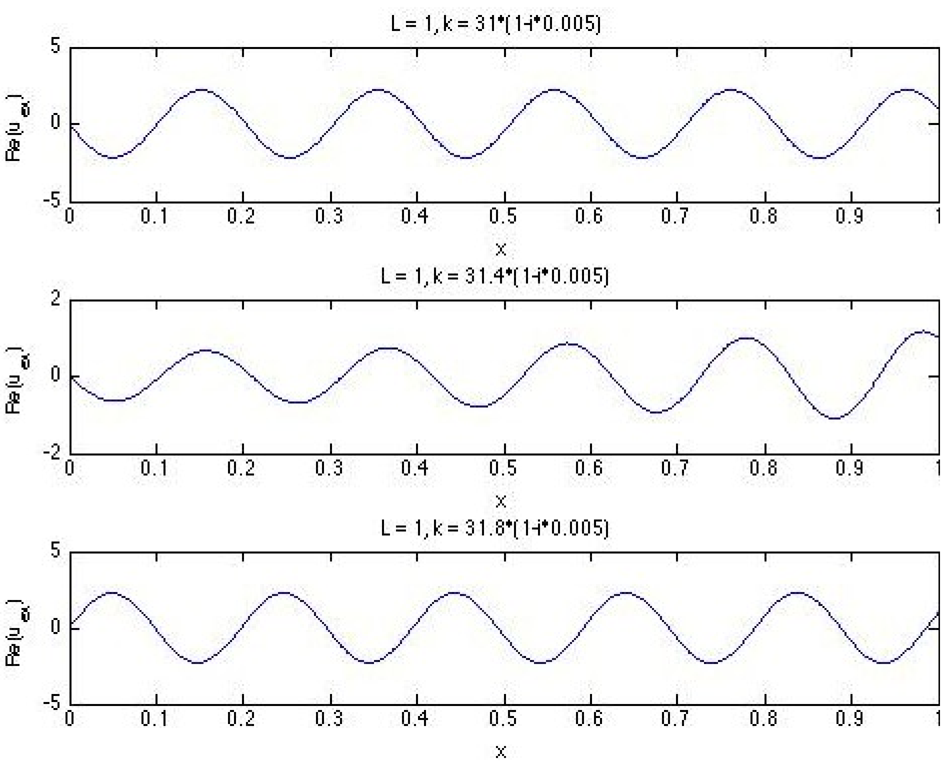


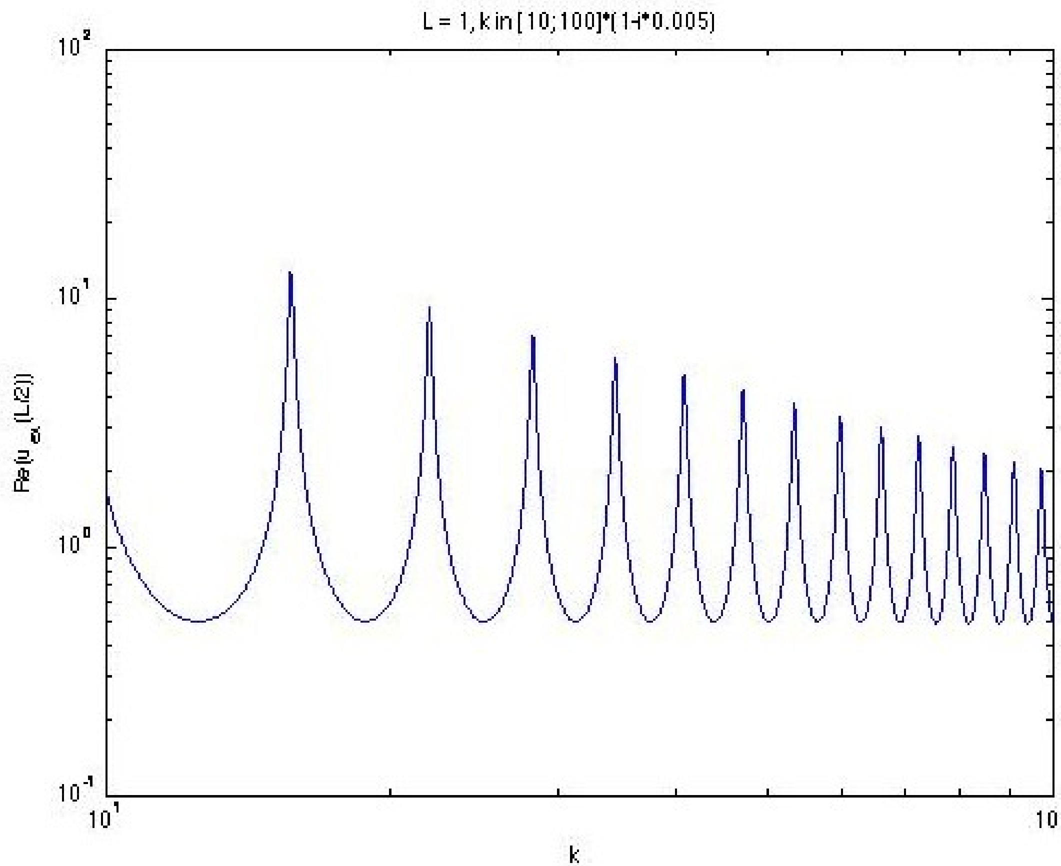


Traditional resolution of DPE

Use of damping : $k = k_0(1 - i\eta)$ $\quad \eta > 0$ (decreasing propagative waves)

No more resonance





Traditional resolution of DPE

FEM approximation

$$\begin{cases} \dfrac{d^2u}{dx^2}+k^2u=0 \quad \text{in } [0;L] \\ u(x=0)=0 \\ u(x=L)=u_L \end{cases} \quad \Leftrightarrow \quad \begin{cases} \text{Find } u\in U \text{ such that} \\ \int_{x=0}^{L}\left(\dfrac{d^2u}{dx^2}+k^2u\right)v\,dx=0 \quad \forall v\in V \\ U=\{u \ / \ u(0)=0 \text{ and } u(L)=u_L\} \\ V=\{v \ / \ v(0)=0 \text{ and } v(L)=0\} \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \text{Find } u\in U \text{ such that} \\ -\int_{x=0}^{L}\dfrac{du}{dx}\dfrac{dv}{dx}dx+k^2\int_{x=0}^{L}u\,v\,dx=0 \quad \forall v\in V \\ U=\{u \ / \ u(0)=0 \text{ and } u(L)=u_L\} \\ V=\{v \ / \ v(0)=0 \text{ and } v(L)=0\} \end{cases}$$

Approximation $u\in U^h\subset U \quad v\in V^h\subset V$

Isoparametric functions $u=\sum_{i=1}^{N}u_i\varphi_i(x) \quad v=\sum_{i=1}^{N}v_i\varphi_i(x)$

$\varphi_i(x)$ set of good functions defining the unknown and the geometry

Traditional resolution of DPE

FEM approximation

$$\left\{\begin{array}{l} \text{Find } u \in U \text{ such that} \\ -\int_{x=0}^{L} \frac{du}{dx}\frac{dv}{dx}dx + k^2\int_{x=0}^{L} u\,v\,dx = 0 \quad \forall v \in V \\ U = \{u \ / \ u(0)=0 \text{ and } u(L)=u_L\} \\ V = \{v \ / \ v(0)=0 \text{ and } v(L)=0\} \end{array}\right.$$

$$\left\{\begin{array}{l} u = \sum_{i=1}^{N} u_i \varphi_i(x) \\ v = \sum_{i=1}^{N} v_i \varphi_i(x) \end{array}\right. \quad \Rightarrow \quad u_0 = 0 \quad u_N = u_L \quad v_0 = 0 \quad v_N = 0$$

$$-\int_{x=0}^{L} \frac{du}{dx}\frac{dv}{dx}dx + k^2\int_{x=0}^{L} u\,v\,dx = 0 \quad \forall \mathbf{V} \in V \quad \Rightarrow \quad \mathbf{V}^T \mathbf{D}\,\mathbf{U} = 0 \quad \forall \mathbf{V} \in V$$

$$\mathbf{D}_{i,j} = -\int_{x=0}^{L} \varphi_i'(x)\varphi_j'(x)dx + k^2\int_{x=0}^{L} \varphi_i(x)\varphi_j(x)dx$$

Traditional resolution of DPE

FEM approximation

$$\left\{\begin{array}{l}\text{Find } u\in U \text{ such that} \\ -\int_{x=0}^{L}\frac{du}{dx}\frac{dv}{dx}dx+k^2\int_{x=0}^{L}u\,v\,dx=0 \quad \forall v\in V \\ U=\{u \ / \ u(0)=0 \text{ and } u(L)=u_L\} \\ V=\{v \ / \ v(0)=0 \text{ and } v(L)=0\}\end{array}\right.$$

$$\left\{\begin{array}{l} u=\sum_{i=1}^{N}u_i\varphi_i(x) \\ v=\sum_{i=1}^{N}v_i\varphi_i(x)\end{array}\right. \Rightarrow \boxed{u_0=0 \quad u_N=u_L \quad v_0=0 \quad v_N=0}$$

$$-\int_{x=0}^{L}\frac{du}{dx}\frac{dv}{dx}dx+k^2\int_{x=0}^{L}u\,v\,dx=0 \quad \forall \mathbf{V}\in V \Rightarrow \boxed{\mathbf{V}^T\mathbf{D}\mathbf{U}=0 \quad \forall \mathbf{V}\in V}$$

$$\mathbf{D}_{i,j}=-\int_{x=0}^{L}\varphi_i'(x)\varphi_j'(x)dx+k^2\int_{x=0}^{L}\varphi_i(x)\varphi_j(x)dx$$

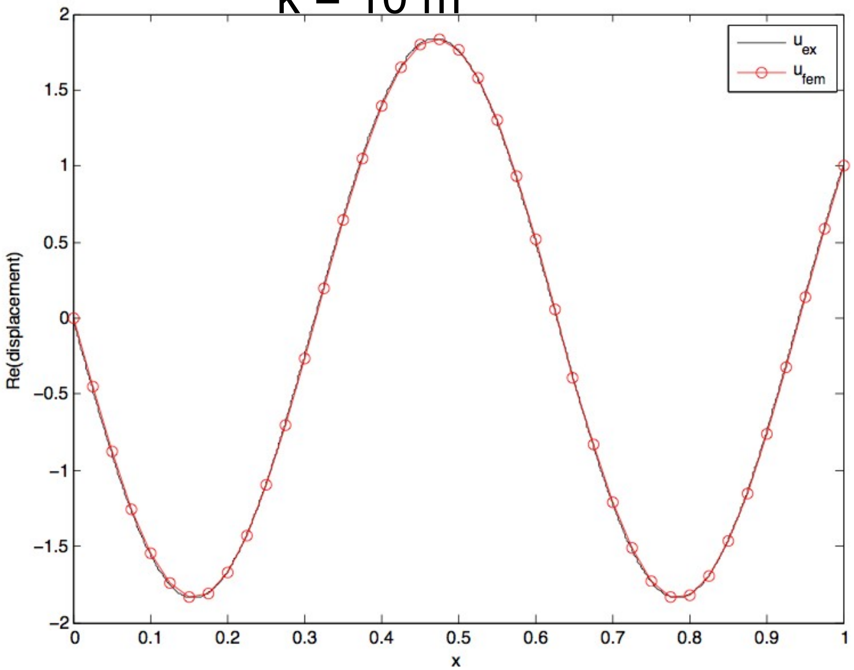
Traditional resolution of DPE

FEM approximation

$$u_0 = 0 \quad u_N = u_L \quad v_0 = 0 \quad v_N = 0 \qquad \Rightarrow \qquad \mathbf{U} = \begin{pmatrix} 0 \\ \tilde{\mathbf{U}} \\ u_L \end{pmatrix} \qquad \mathbf{V} = \begin{pmatrix} 0 \\ \tilde{\mathbf{V}} \\ 0 \end{pmatrix}$$

$$-\mathbf{V}'\mathbf{D}\mathbf{U} = 0 \quad \forall \mathbf{V} \in V \qquad \Rightarrow \tilde{\mathbf{V}}\left(\tilde{\mathbf{D}}\tilde{\mathbf{U}} + \mathbf{D}_{2:N-1;N} u_L\right) = 0 \quad \forall \tilde{\mathbf{V}} \qquad \Rightarrow \tilde{\mathbf{D}}\tilde{\mathbf{U}} = -\mathbf{D}_{2:N-1;N} u_L$$

k = 10 m^{-1}



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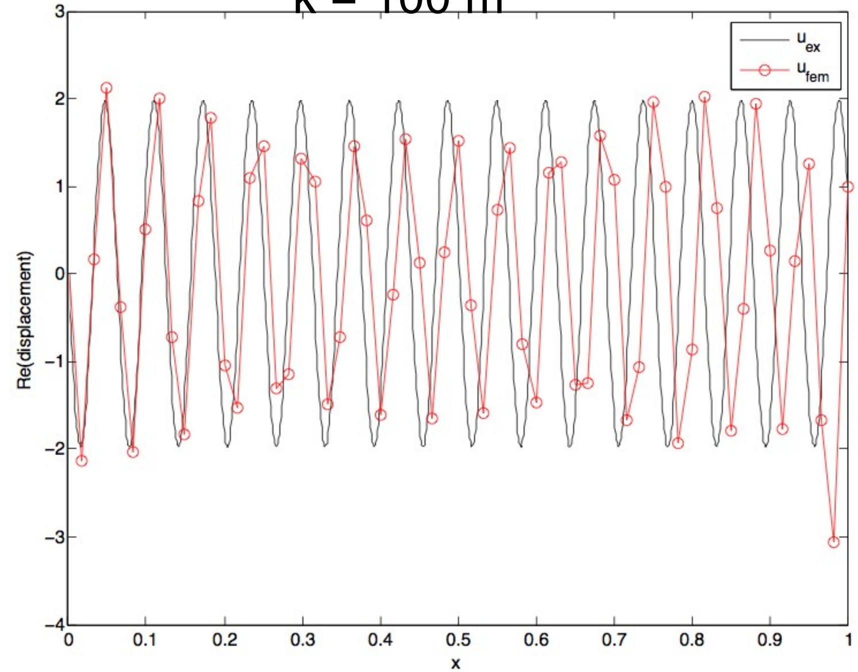
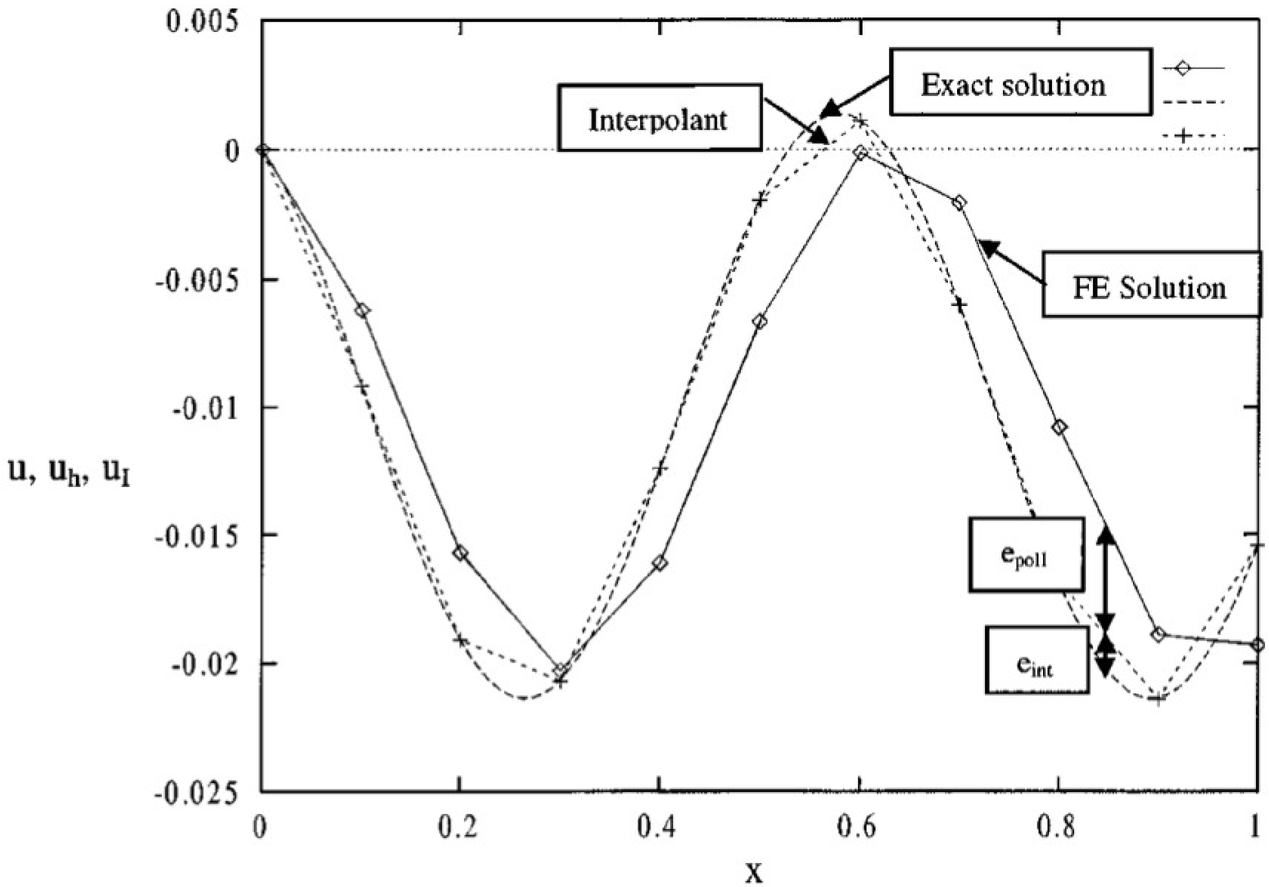


Traditional resolution of DPE

FEM approximation



Deraemaeker et al. 1999

Traditional resolution of DPE

FEM approximation

Ihlenburg et Babushka 1997

$$\frac{|u^h - u_{ex}|_1}{|u^h|_1} < C_1 \left(\frac{k\,h}{p}\right)^p + C_2\, k\, L \left(\frac{k\,h}{p}\right)^{2p}$$

Interpolation error

Pollution error

=> FEM (piecewise polynomial functions) not useful in midfrequency due to the increasing number of dofs.

=> Origin: try to approximate waves by polynomial functions

Treffz method

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Treffz method

Governing equation

$$\begin{cases} \dfrac{d^2u}{dx^2}+k^2u=0 \ \text{ in } [0;L] \\ u(x=0)=0 \\ u(x=L)=u_L \end{cases}$$

Test function which satisfies the governing equation

$$\varphi_1(x)=e^{ikx} \qquad \varphi_2(x)=e^{-ikx}$$

$$u(x)=u_1\varphi_1(x)+u_2\varphi_2(x)=u_1e^{ikx}+u_2e^{-ikx}$$

Advantage	Drawback
Few dofs Direct link with physics Take into account the rapid scale	Difficulty to find test functions Need for approximations for boundary condition Numerical difficulties may appear

Treffz method

Governing equation

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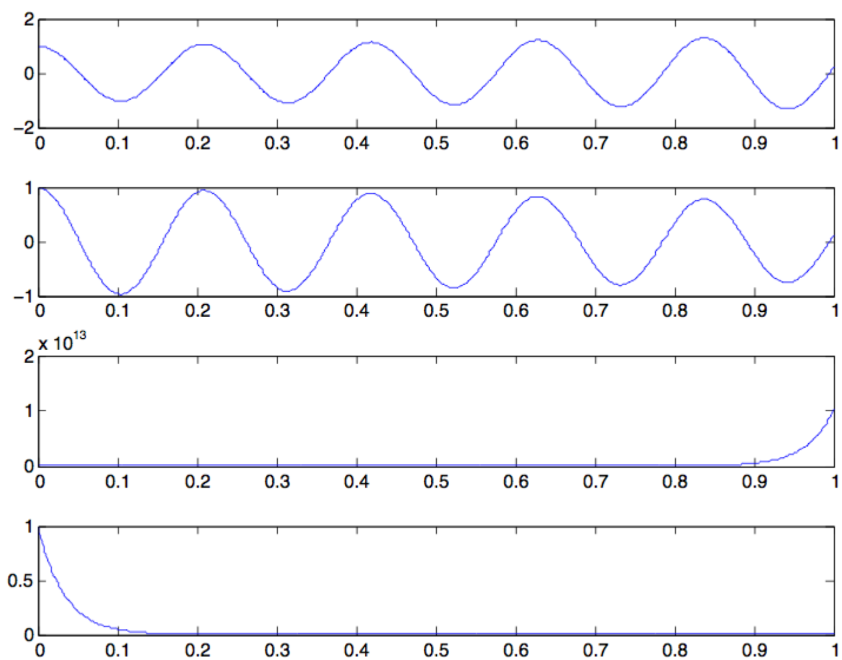
Treffz method

Difficulty to find test functions - 1D

$$\frac{d^2u}{dx^2}+k^2u=0 \quad \text{(bars)} \quad \Rightarrow \quad \varphi_1(x)=e^{ikx} \quad \varphi_2(x)=e^{-ikx}$$

$$\frac{d^4u}{dx^4}-k^4u=0 \quad \text{(beams)} \quad \Rightarrow \quad \varphi_1(x)=e^{ikx} \quad \varphi_2(x)=e^{-ikx} \quad \varphi_3(x)=e^{kx} \quad \varphi_4(x)=e^{-kx}$$



$\varphi_1(x)=e^{ikx}$

$\varphi_2(x)=e^{-ikx}$

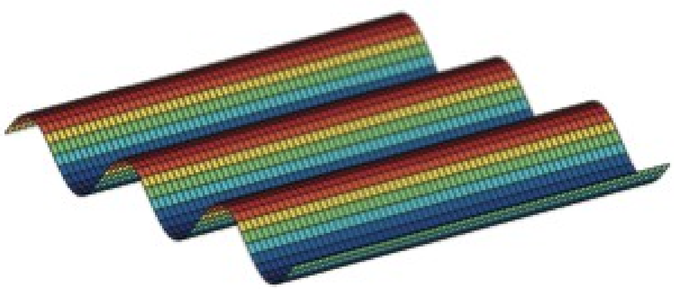
$\varphi_3(x)=e^{kx}$ → Complex computation may appear

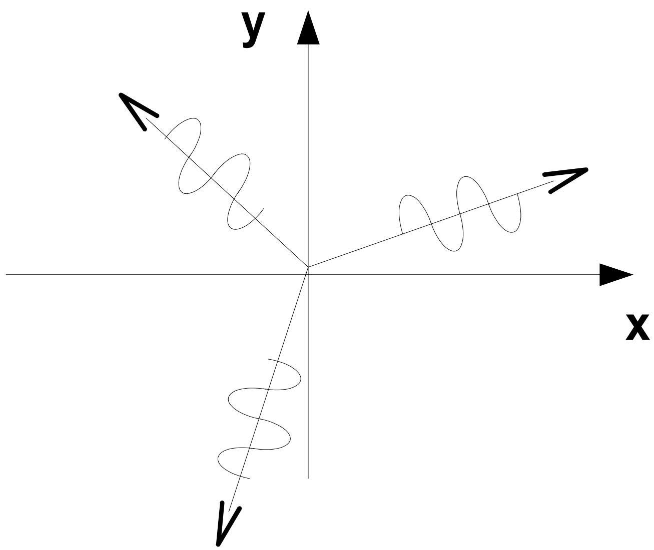
=> $\tilde{\varphi}_3(x)=e^{k(x-L)}$

$\varphi_4(x)=e^{-kx}$

Difficulty to find test functions - 2D

$\Delta u + k^2 u = 0$ (acoustics) $\Rightarrow$ $\varphi_n(x) = e^{i \mathbf{k_n} \mathbf{x}}$ $\mathbf{k_n} = k \cos\theta_n \mathbf{x} + k \sin\theta_n \mathbf{y}$





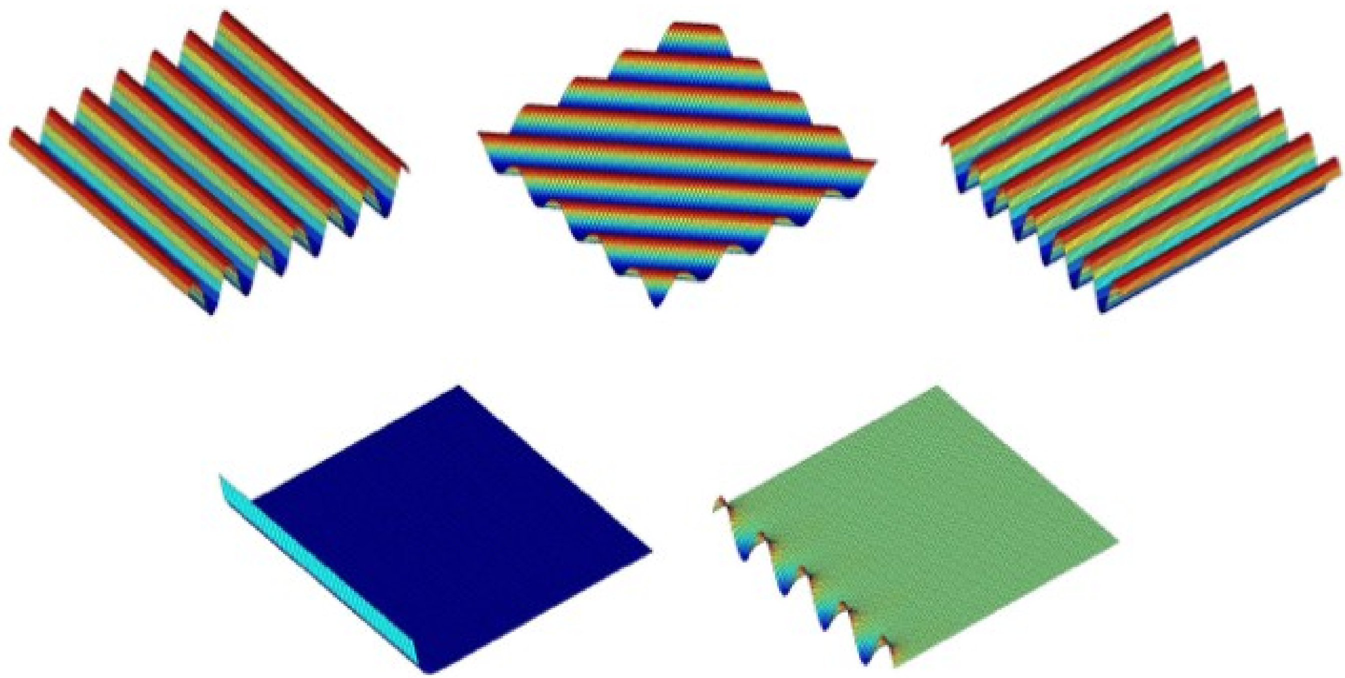
Treffz method

Difficulty to find test functions - 2D

$\Delta\Delta u - k^4 u = 0$ (out of plane flexure) →

$\varphi_n(x) = e^{i\mathbf{k}_n \mathbf{x}}$ $\mathbf{k}_n = k\cos\theta_n \mathbf{x} + k\sin\theta_n \mathbf{y}$ (propagative)

$\varphi_m(x) = e^{\mathbf{k}_m \mathbf{x}}$ $\mathbf{k}_n = k\sqrt{1+\cos^2\theta_m}\,\mathbf{x} + ik\sin\theta_m \mathbf{y}$ (evanescent)



Treffz method

Difficulty to find test functions - 2D

$$\mathbf{div\,N} - \mathbf{B}\,\mathbf{div\,M} = -\rho\,\omega^2 h\,\mathbf{v}$$

$$\mathrm{div}\,\mathbf{div\,M} + \mathrm{Tr}(\mathbf{NB}) = -\rho\,\omega^2 h\,w$$

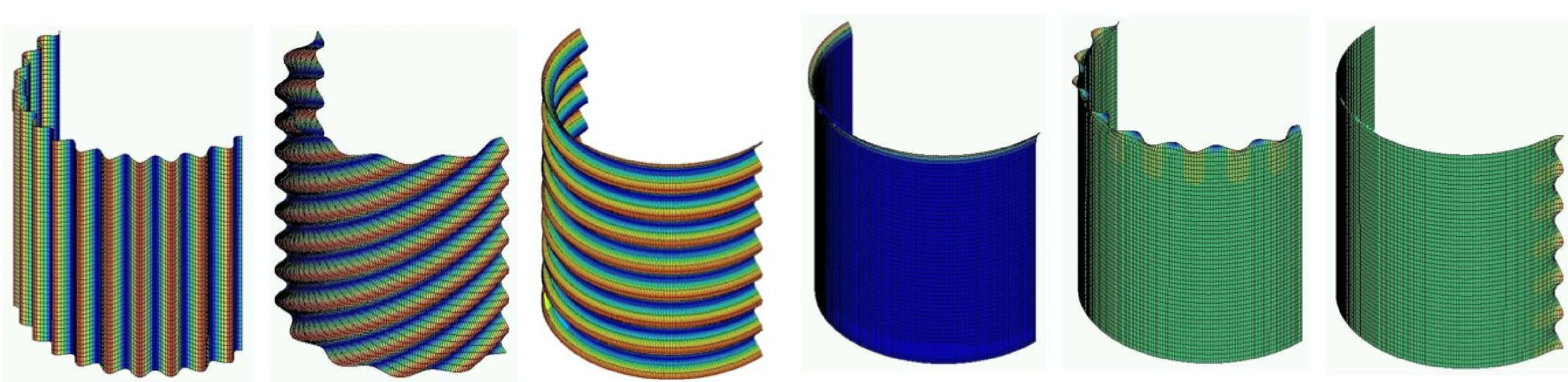
$$\mathbf{M} = \frac{h^3}{12}\mathbf{K}_{CP}\,\mathbf{X}(\mathbf{u})$$

$$\mathbf{N} = h\,\mathbf{K}_{CP}\,\boldsymbol{\gamma}(\mathbf{u})$$

(shell)

$$\varphi_n(x) = e^{i\mathbf{k}_n\mathbf{x}}$$

$$(\mathbf{k}_n.\mathbf{k}_n)^4 = \frac{12(1-\nu^2)\rho\,\omega^2}{Eh^2}(\mathbf{k}_n\mathbf{k}_n)^2 + \frac{12(1-\nu^2)}{h^2}(\mathbf{k}_n.\mathbf{R}.\mathbf{B}.\mathbf{R}.\mathbf{k}_n)^2$$

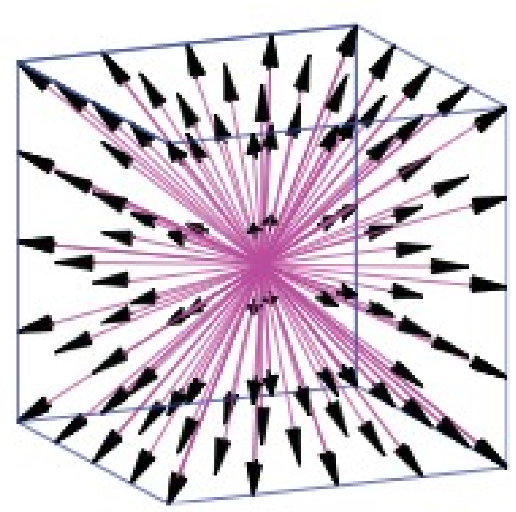


(propagative) (evanescent)

Treffz method

Difficulty to find test functions - 3D

$\Delta u + k^2 u = 0$ (3D acoustics) → $\varphi_n(x) = e^{i\mathbf{k}(\theta_n, \varphi_n)\mathbf{x}}$



Treffz method

Difficulty to find test functions

Other functions available:

$\varphi_n(x)=e^{i\mathbf{k}_n\mathbf{x}}$ test function → $\phi_n(x)=\int_{\theta\in\Theta} e^{i\mathbf{k}_n(\theta)\mathbf{x}}\,d\theta$ test function (wave band)

$\varphi_n(x)=e^{i\mathbf{k}_n\mathbf{x}}$ test function → $\phi_n(x)=\int_{\theta\in[0;2\pi]} f(\theta)e^{i\mathbf{k}_n(\theta)\mathbf{x}}\,d\theta$ test function

Vekua functions $\varphi_n(x)=e^{in\psi}J_n(kr)$ $\quad r=\sqrt{x^2+y^2}$ $\quad \psi=\tan^{-1}y/x$

$\varphi_n(x)=\cos(k_{nx}x)e^{-ik_{ny}y}$ $\quad k_{nx}=\frac{n\pi}{L}$ $\quad k_{ny}=\pm\sqrt{k^2-\left(\frac{n\pi}{L}\right)}$

...

Mandatory: space of shape function able to span all the solutions

For 3D see *[Kovalevsky et al. 12]*

Treffz method

Governing equation

$$\begin{cases} \dfrac{d^2u}{dx^2}+k^2u=0 \text{ in } [0;L] \\ u(x=0)=0 \\ u(x=L)=u_L \end{cases}$$

Test function which satisfies the governing equation

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$$u(x)=u_1\varphi_1(x)+u_2\varphi_2(x)=u_1e^{ikx}+u_2e^{-ikx}$$

Advantage	Drawback
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Treffz method

$$\left.\begin{aligned} u(x=0)&=0 \\ u(x=L)&=u_L \end{aligned}\right\} \text{ Approximation ?}$$

Least square: find $u \in U$ in order to mnimize

$$\alpha |u(0)|^2 + \beta |u(L) - u_L|^2$$

Hybrid formulation: find $u \in U, \lambda \in W$ such that

$$a(u,v) - <\lambda, v> = L(v) \quad \forall v \in V$$

$$<\mu, u> = L_b(\mu) \quad \forall \mu \in W$$

Dedicated variational formulation:

$$\text{find} u \in U \text{ such that}$$

$$-u(0)\frac{dv^*}{df}(0) + (u(L) - u_L)\frac{dv^*}{df}(L) = 0 \, \forall v \in V$$

....

See Ladevèze 1995, Melenk et Babushka 1996, Cessenat 1996, Desmet 1998, Monk et Wang 1999, Laghrouche et Bettess 2000, Farhat et al. 2001, Strouboulis et al. 2008, Gittelson et al. 2009, Hiptmair et al. 2011 , ...

Treffz method

Dedicated variational formulation: *[Ladevèze 1995]*

$$\text{find } u \in U \text{ such that}$$

$$-u(0)\frac{dv^*}{df}(0)+(u(L)-u_L)\frac{dv^*}{df}(L)=0 \quad \forall v \in V$$

Unicity proof: imagine two solutions u_1 and u_2 in U, and let us note w (in V) their difference. We then have

$$-w(0)\frac{dw^*}{dx}(0)+w(L)\frac{dw^*}{dx}(L)=0$$

but

$$-w(0)\frac{dw^*}{dx}(0)+w(L)\frac{dw^*}{dx}(L)=0 \;\Rightarrow\; \left[w(x)\frac{dw^*}{dx}(x)\right]_0^L=0 \;\Rightarrow\; \int_{x=0}^{L}\frac{dw}{dx}\frac{dw^*}{dx}dx+\int_{x=0}^{L}w\frac{d^2w^*}{dx^2}dx=0$$

remembering $u \in U \Rightarrow \frac{d^2u}{dx^2}+k^2u=0$ we have

$$\int_{x=0}^{L}\frac{dw}{dx}\frac{dw^*}{dx}dx-\int_{x=0}^{L}w\,k^{*2}w^*dx=0$$

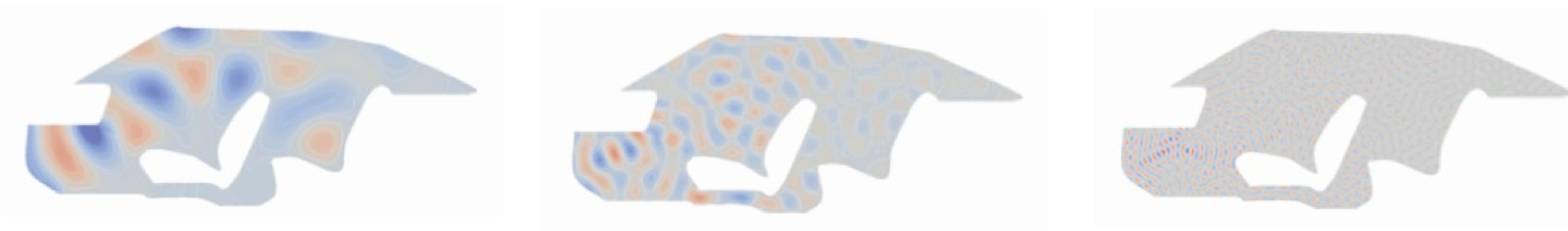
with $k=k_0(1-i\eta)$, the imaginary part gives $-2\eta k_0^2\int_{x=0}^{L}w\,w^*dx=0$

(in relation to the dissipated power), then w = 0,

then the solution is unique, and equal to the reference solution.

Mid frequencies

Frequency increase



Element methods:
FEM, BEM, ...

Wave methods:
VTCR, WBM, DEM, PUM, LSM, ...

Energy based methods:
SEA, SmEDA, WIA, ...

Limited due to fine discretization

Limited by high frequency hypothesis

Morand[92], Mercier [93], Soize [98], Liu and al. [91], Hugues [95], Fleuret [97], Cessenat and Després [98] Harari and Haham [98], Greenstadt [99], Laghrouche and Bettess [99], Farhat and al. [00] Strouboulis [06], De Langre [91], Perrey Debain and al. [03]

Ladevèze [96], Cessenat and Despres [98], Desmet [98], Monk and Wang [99], Farhat [01], Perrey Debain and al. [04], T. Strouboulis and R. Hidajat [06], Gittelson and al. [09], Hiptmair and al. [11]

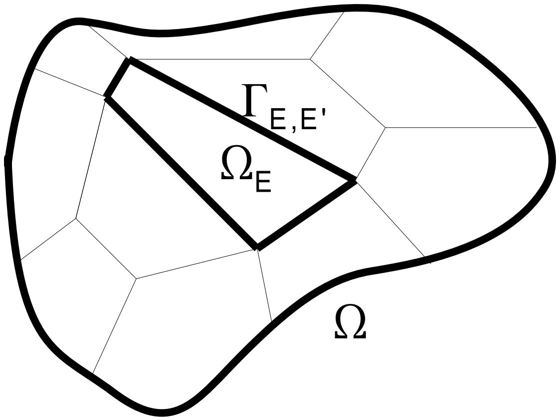
Lyon and Maidanik [62], Belov and Ryback [75], Nefske and Sung [89], Ichchou and al. [97], Le Bot [98], Krokstadt [98], Maxit and Guyader [01], Chae and Ih [01], Langley [92], Cotoni and Langley [04], Ichchou and al. [09], Totaro and Guyader [12], Savin [13]

The Variational Theory of Complex Rays

[Ladevèze 96]

- Reference problem (acoustics)

$$\begin{cases} \text{Find } u=\{u_E\}_{E\in \mathbf{E}} \text{ such that} \\ \Delta u + k^2 u = 0 \ \text{ in } \ \Omega_E \\ \mathbf{q_u}.\mathbf{n} + Z u = g_d \ \text{ on } \ \partial\Omega \\ \{\mathbf{q_u}.\mathbf{n}\}_{E,E'} = 0 \text{ and } [u]_{E,E'} = 0 \ \text{ on } \ \Gamma_{E,E'} \end{cases}$$



$$\mathbf{q_u} = \mathbf{grad}\, u$$

$$\{u\}_{E,E'} = (u_E + u_{E'})_{\Gamma_{E,E'}}$$

$$[u]_{E,E'} = (u_E - u_{E'})_{\Gamma_{E,E'}}$$

- Variational formulation

$$\begin{cases} \text{Find } u \in U \text{ such that} \\ a(u,v) = l(v) \ \ \forall v \in U_0 \end{cases}$$

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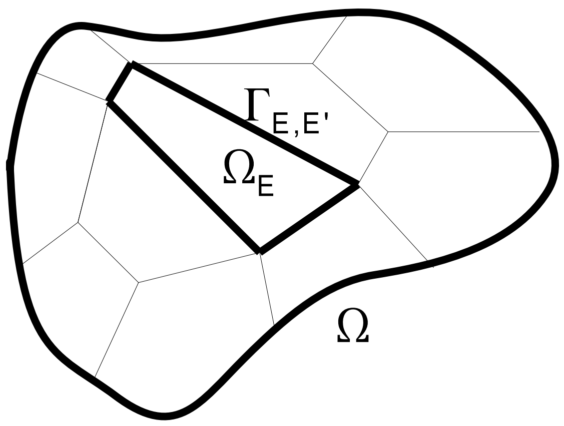

- Variational formulation

$$\begin{cases} \text{Find } u \in U \text{ such that} \\ a(u,v) = l(v) \quad \forall v \in U_0 \end{cases}$$

$U = \cup_E U_E$ is the space of functions which verify the equilibrium and constitutive relation (U_0 is the associated homogeneous space) => Trefftz method

The approximations are independent from one substructure to another $\Rightarrow$ Flexibility and efficiency of the method

The Variational Theory of Complex Rays

[Ladevèze 96]

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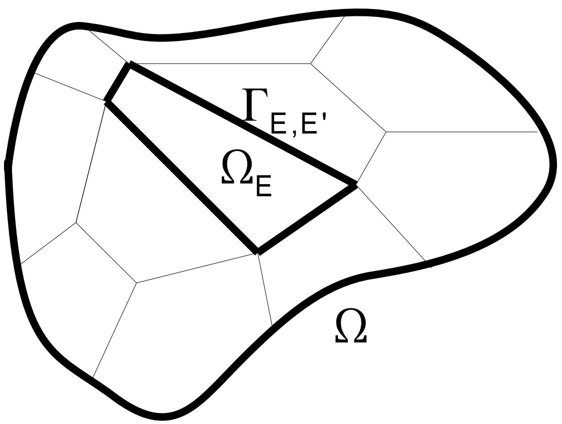


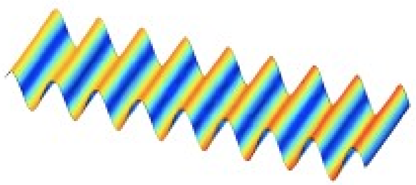
- Variational formulation

$$\begin{cases} \text{Find } u \in U \text{ such that} \\ a(u,v) = l(v) \quad \forall v \in U_0 \end{cases}$$

For acoustics,

One has to verify $\Delta u_E + k_E^2 u_E = 0$

Solutions are waves



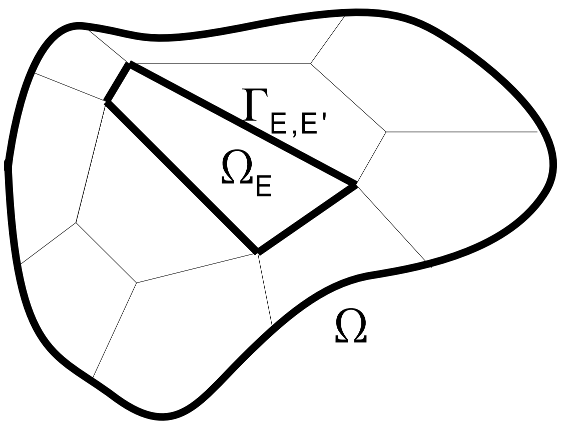
Propagative waves

The Variational Theory of Complex Rays

[Ladevèze 96]

- Reference problem (acoustics)

$$\begin{cases} \text{Find } u = \{u_E\}_{E \in \mathbf{E}} \text{ such that} \\ \Delta u + k^2 u = 0 \text{ in } \Omega_E \\ \mathbf{q_u}.\mathbf{n} + Zu = g_d \text{ on } \partial\Omega \\ \{\mathbf{q_u}.\mathbf{n}\}_{E,E'} = 0 \text{ and } [u]_{E,E'} = 0 \text{ on } \Gamma_{E,E'} \end{cases}$$



- Variational formulation

$$\begin{cases} \text{Find } u \in U \text{ such that} \\ a(u,v) = l(v) \quad \forall v \in U_0 \end{cases}$$

a(.,.) et l(.) are bilinear and linear forms equivalent to boundary conditions and interface conditions

The Variational Theory of Complex Rays

[Ladevèze 96]

- Reference problem (acoustics)

$$\begin{cases} \text{Find } u=\{u_E\}_{E\in \mathbf{E}} \text{ such that} \\ \Delta u + k^2 u = 0 \text{ in } \Omega_E \\ \mathbf{q_u}.\mathbf{n} + Zu = g_d \text{ on } \partial\Omega \\ \{\mathbf{q_u}.\mathbf{n}\}_{E,E'} = 0 \text{ and } [u]_{E,E'} = 0 \text{ on } \Gamma_{E,E'} \end{cases}$$

$$\sum_{E,E'} \int_{\Gamma_{E,E'}} \left(\frac{1}{2} \{\mathbf{q_u}.\mathbf{n}\}_{E,E'} \{\tilde{v}\}_{E,E'} - \frac{1}{2} [\tilde{\mathbf{q}}_\mathbf{v}.\mathbf{n}]_{E,E'} [u]_{E,E'} \right) dS + \sum_E \int_{\partial\Omega} (\mathbf{q_u}.\mathbf{n} + Zu - g_d) \tilde{v} \, dS = 0 \quad \forall v \in U_0$$

- Variational formulation

$$\begin{cases} \text{Find } u \in U \text{ such that} \\ a(u,v) = l(v) \quad \forall v \in U_0 \end{cases}$$

a(.,.) et l(.) are bilinear and linear forms equivalent to boundary conditions and interface conditions

The Variational Theory of Complex Rays

[Ladevèze 96]

- Reference problem (acoustics)

Find $u=\{u_E\}_{E\in\mathbf{E}}$ such that

$$\begin{cases} \Delta u + k^2 u = 0 \text{ in } \Omega_E \\ \mathbf{q}_u.\mathbf{n} + Zu = g_d \text{ on } \partial\Omega \\ \{\mathbf{q}_u.\mathbf{n}\}_{E,E'} = 0 \text{ and } [u]_{E,E'} = 0 \text{ on } \Gamma_{E,E'} \end{cases}$$

- Variational formulation

Find $u \in U$ such that

$$a(u,v) = l(v) \quad \forall v \in U_0$$

$$\sum_{E,E'} \int_{\Gamma_{E,E'}} \left(\frac{1}{2}\{\mathbf{q}_u.\mathbf{n}\}_{E,E'}\{\tilde{v}\}_{E,E'} - \frac{1}{2}[\tilde{\mathbf{q}}_v.\mathbf{n}]_{E,E'}[u]_{E,E'} \right) dS$$

$$+\sum_E \int_{\partial\Omega} (\mathbf{q}_u.\mathbf{n} + Zu - g_d)\tilde{v}\, dS = 0 \quad \forall v \in U_0$$

Boundary condition

a(.,.) et l(.) are bilinear and linear forms equivalent to boundary conditions and interface conditions

The Variational Theory of Complex Rays

[Ladevèze 96]

- Reference problem (acoustics)

  Find $u=\{u_E\}_{E\in\mathbf{E}}$ such that

  $\Delta u+k^2u=0$ in Ω_E

  $\mathbf{q}_u.\mathbf{n}+Zu=g_d$ on $\partial\Omega$

  $\{\mathbf{q}_u.\mathbf{n}\}_{E,E'}=0$ and $[u]_{E,E'}=0$ on $\Gamma_{E,E'}$

- Variational formulation

  Find $u\in U$ such that

  $a(u,v)=l(v) \quad \forall v\in U_0$

Interface condition

$$\sum_{E,E'}\int_{\Gamma_{E,E'}}\left(\frac{1}{2}\{\mathbf{q}_u.\mathbf{n}\}_{E,E'}\{\tilde{v}\}_{E,E'}-\frac{1}{2}[\tilde{\mathbf{q}}_v.\mathbf{n}]_{E,E'}[u]_{E,E'}\right)dS$$

$$+\sum_E\int_{\partial\Omega}(\mathbf{q}_u.\mathbf{n}+Zu-g_d)\tilde{v}\,dS=0 \quad \forall v\in U_0$$

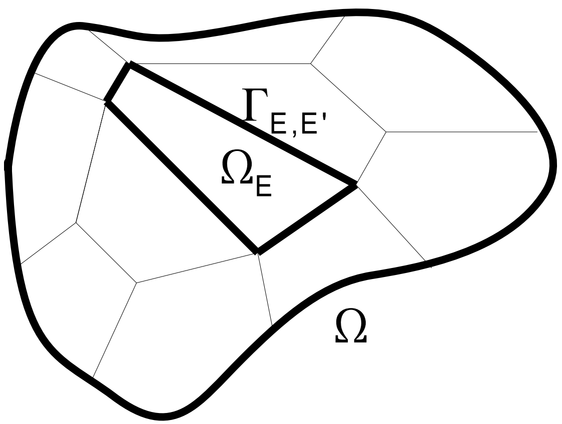
a(.,.) et l(.) are bilinear and linear forms equivalent to boundary conditions and interface conditions

The Variational Theory of Complex Rays

[Ladevèze 96]

- Reference problem (acoustics)

$$\begin{cases} \text{Find } u=\{u_E\}_{E\in \mathbf{E}} \text{ such that} \\ \Delta u + k^2 u = 0 \text{ dans } \Omega_E \\ \mathbf{q_u}.\mathbf{n} + Z u = g_d \text{ sur } \partial\Omega \\ \{\mathbf{q_u}.\mathbf{n}\}_{E,E'}=0 \text{ et } [u]_{E,E'}=0 \text{ sur } \Gamma_{E,E'} \end{cases}$$



- Variational formulation

$$\begin{cases} \text{Find } u\in U \text{ such that} \\ a(u,v)=l(v) \quad \forall v\in U_0 \end{cases}$$

$$u(\mathbf{x})=\int_\theta a(\theta)e^{i\mathbf{k}.\mathbf{x}}d\theta$$

$$\downarrow$$

$$u^h(\mathbf{x})=\sum_{j=1}^{N} a_j \int_{\theta_i}^{\theta_{i+1}} e^{i\mathbf{k}.\mathbf{x}}d\theta$$

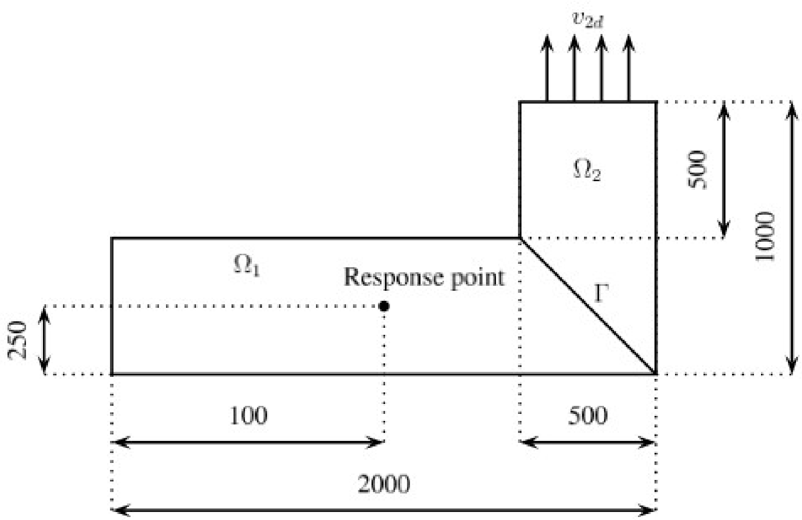
The rapid scale is preserved

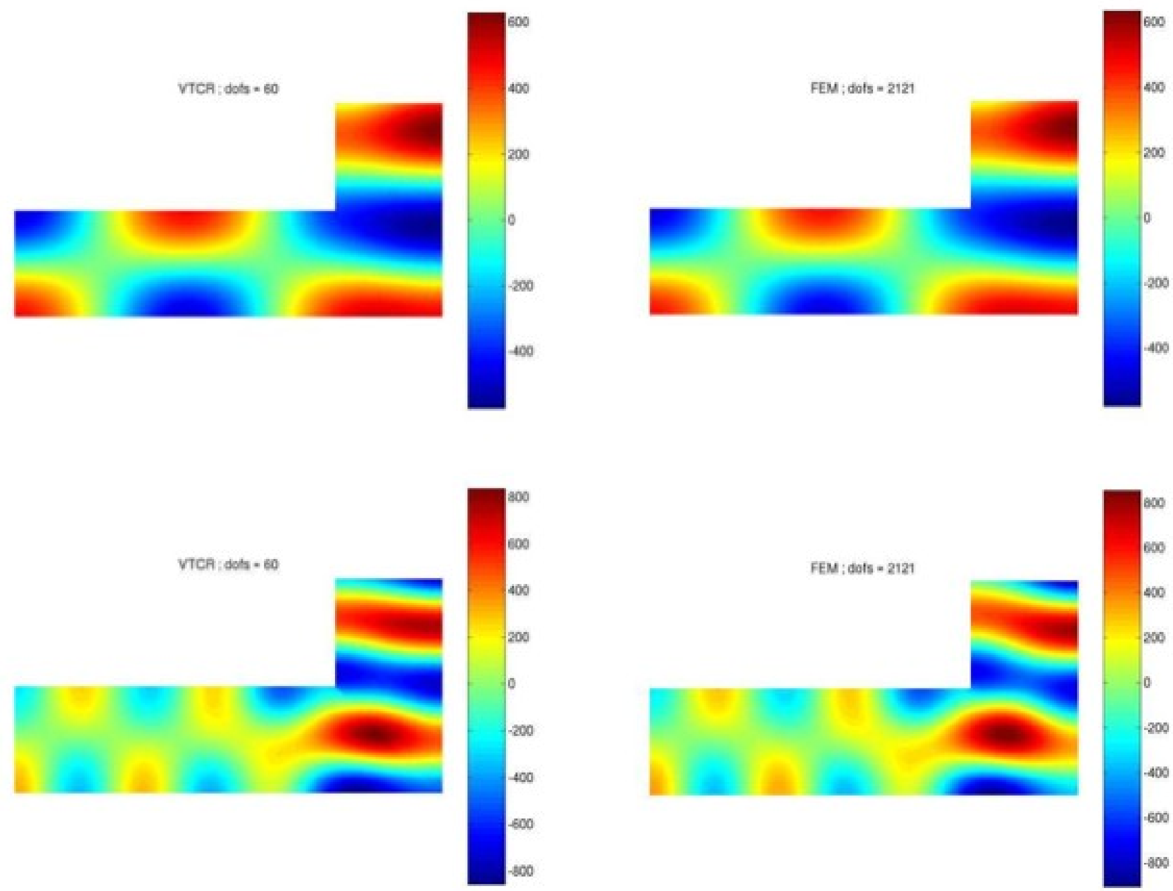
- Approximated solution

$$\begin{cases} U^h\subset U \\ \text{Find } u^h\in U^h \text{ such that} \\ a(u^h,v^h)=l(v^h) \quad \forall v^h\in U_0^h \end{cases}$$

The Variational Theory of Complex Rays

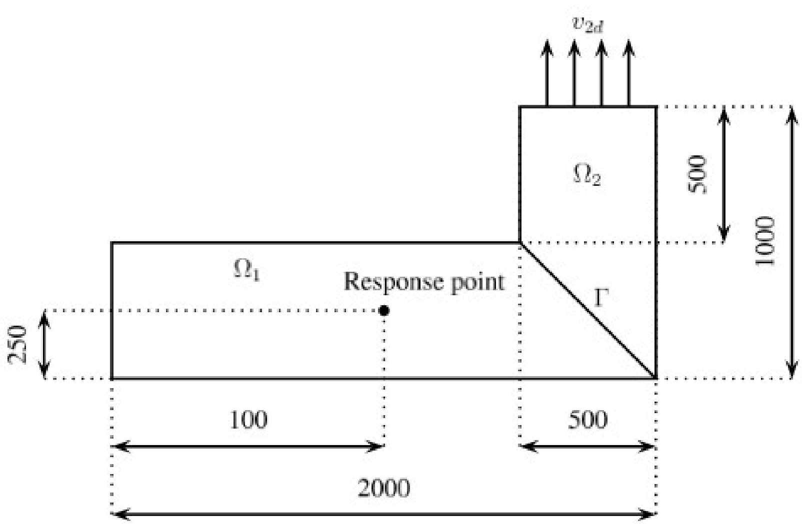
[Riou 08]

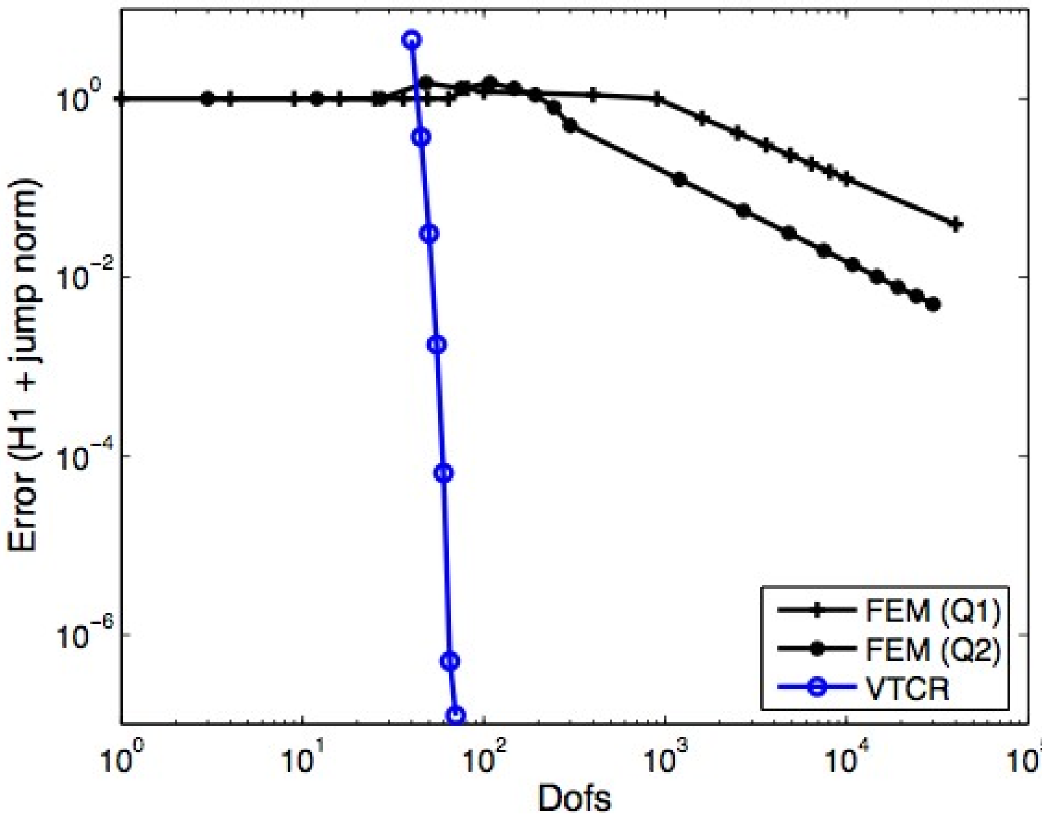




The Variational Theory of Complex Rays

[Riou 08]



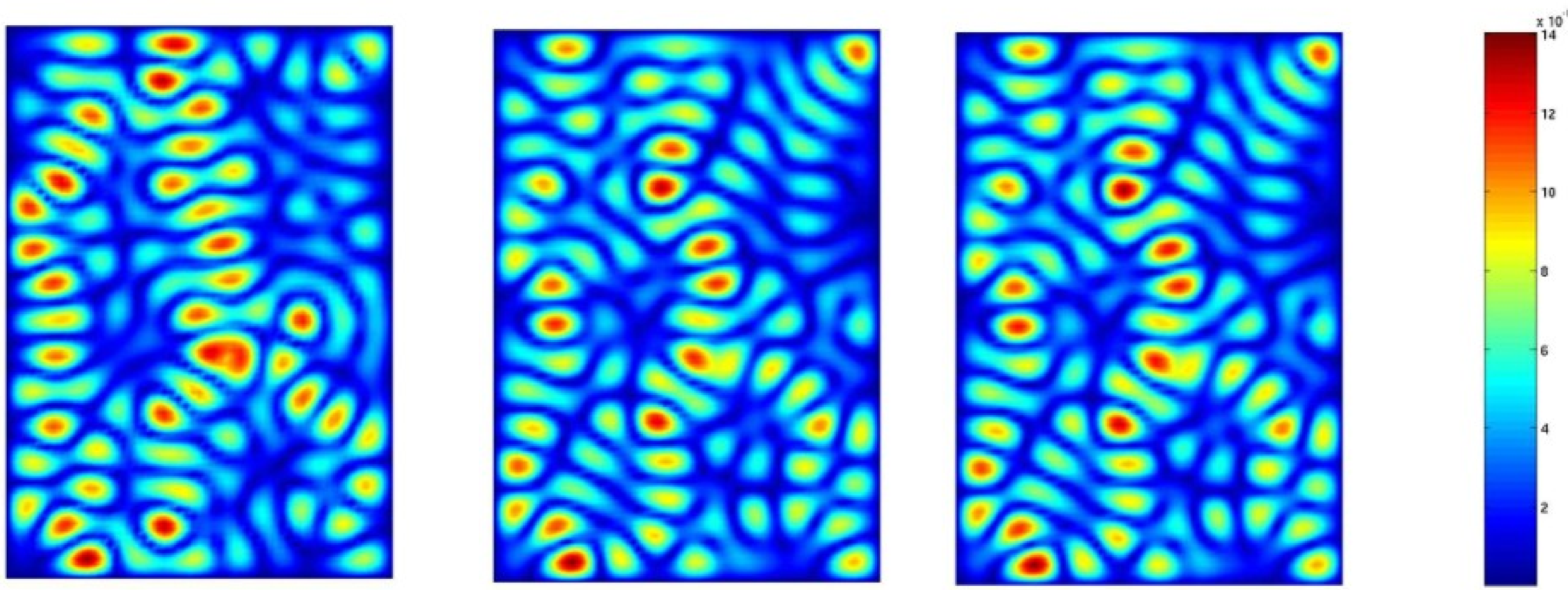


The Variational Theory of Complex Rays

[Riou 04]

- Example: simply supported plate ⟶ Edge waves

  Size: 0.7 m x 1 m. Frequency: 2000 Hz.
  Thickness: 3 mm. Punctual unitary force
  (0.05,−0.1) m. Damping: 0.01.



FEM 39046 DOFs (10 elements in a wavelength)

Analytical solution

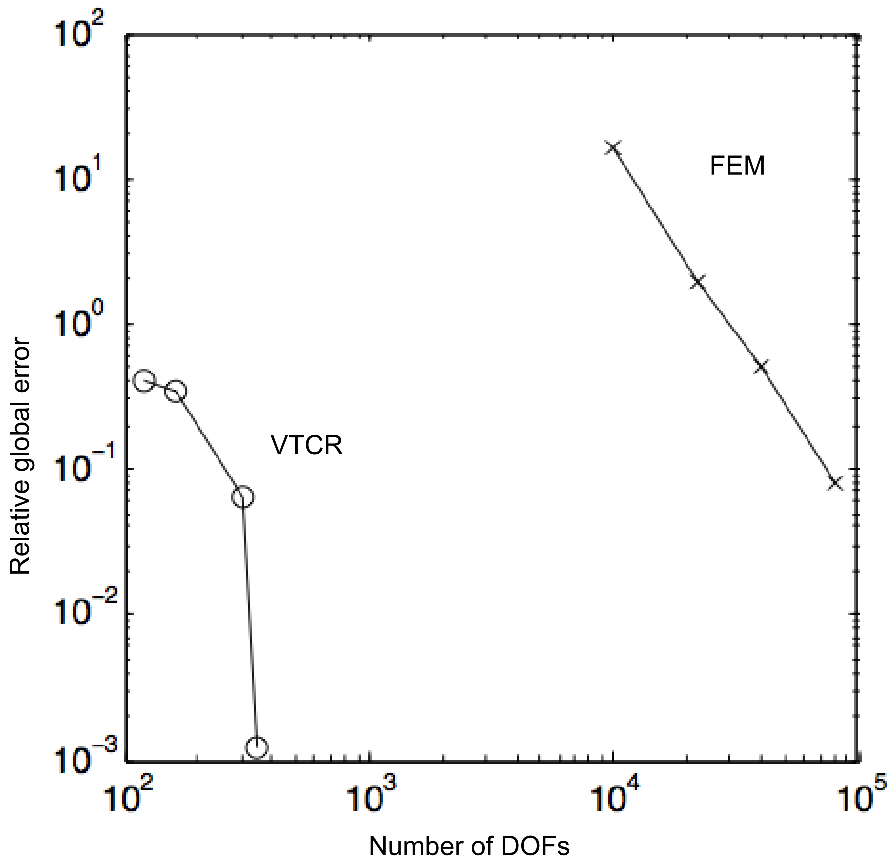
VTCR (180 DOFs)

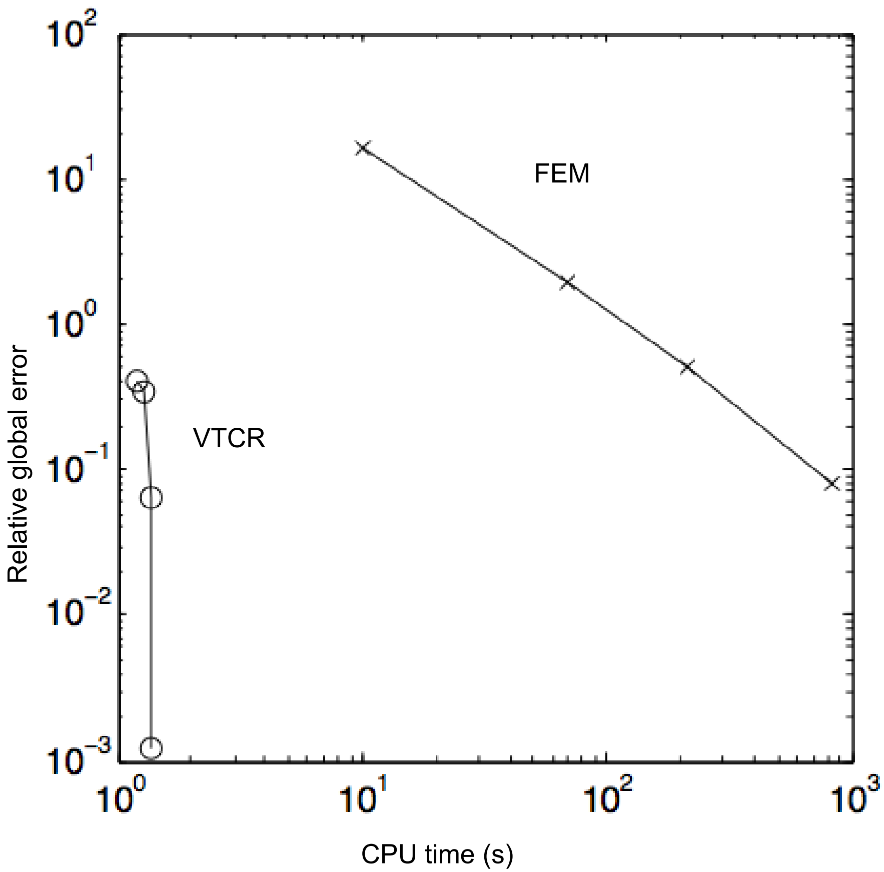
The Variational Theory of Complex Rays

[Riou 04]

- Example: simply supported plate

  Size: 0.7 m x 1 m. Frequency: 2000 Hz.
  Thickness: 3 mm. Punctual unitary force
  (0.05,−0.1) m. Damping: 0.01.

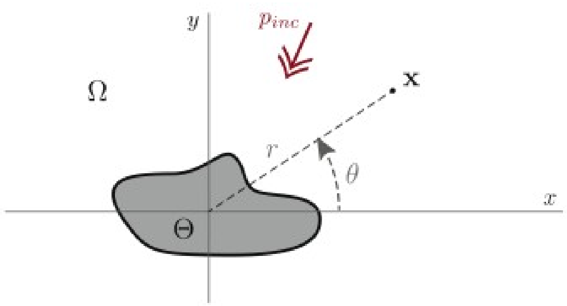




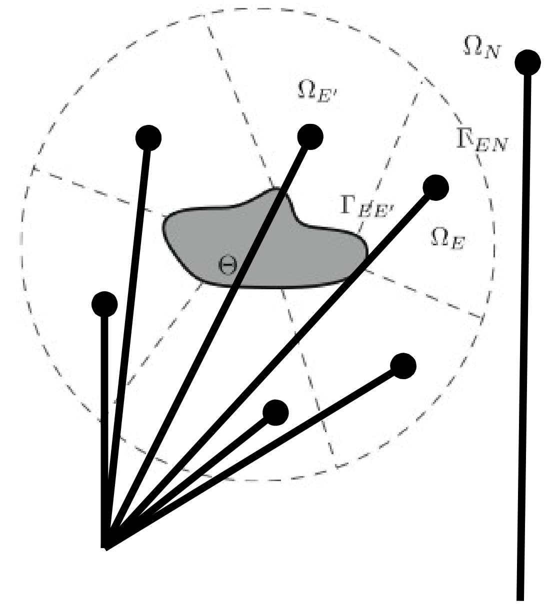
The Variational Theory of Complex Rays

[Kovalevsky 13]

Acoustic scattering



Subdomains



Classic accoustic + Sommerfled

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial p}{\partial r} - ikp = 0\right)$$

Classic Trefftz

Trefftz +Sommerfeld

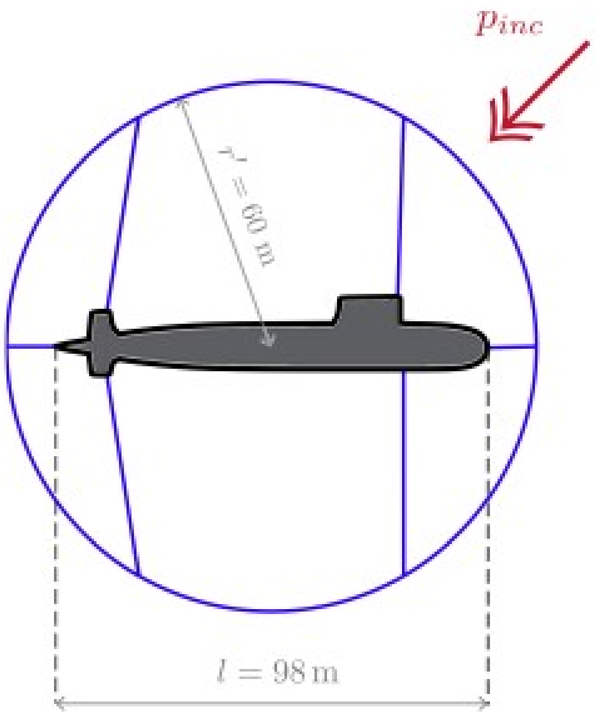
$$p_{\Omega_N} = \sum_{n=-N_N}^{n=+N_N} \alpha_N e^{in\theta} H_{|n|}^{(2)}(kr)$$

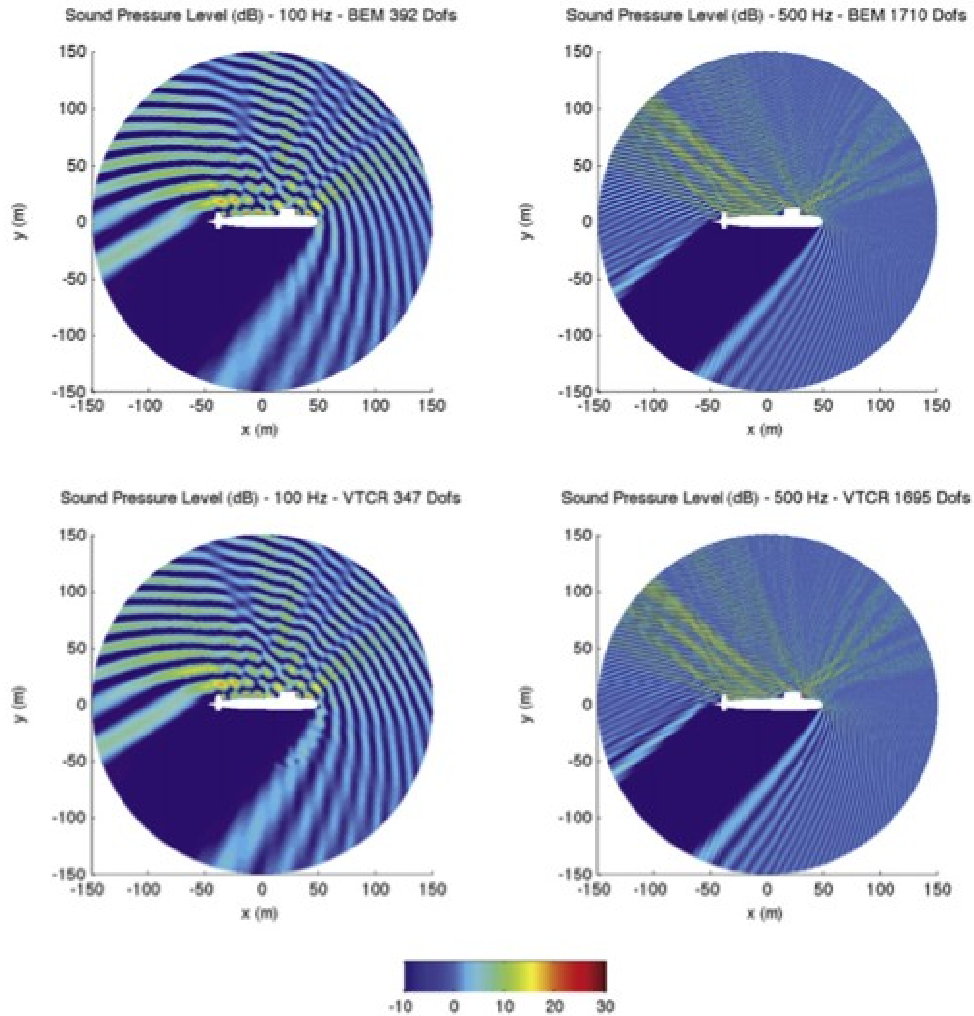
The Variational Theory of Complex Rays

[Kovalevsky 13]

Acoustic scattering

Density: 1.2 kg/m³. Speed: 340 m/s.
Damping: 0.0001.
Frequency: 100 and 500 Hz

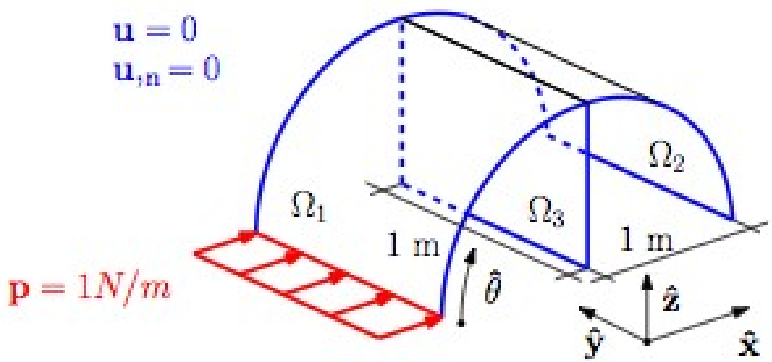




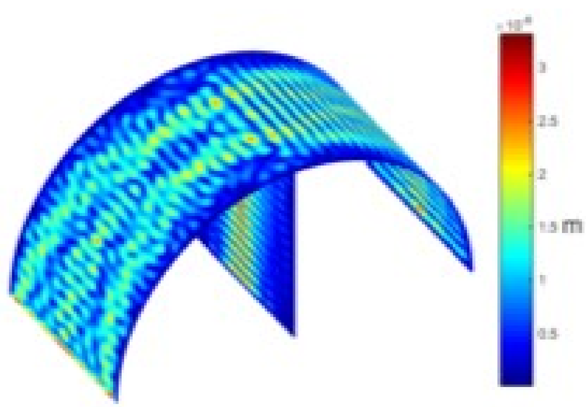
The Variational Theory of Complex Rays

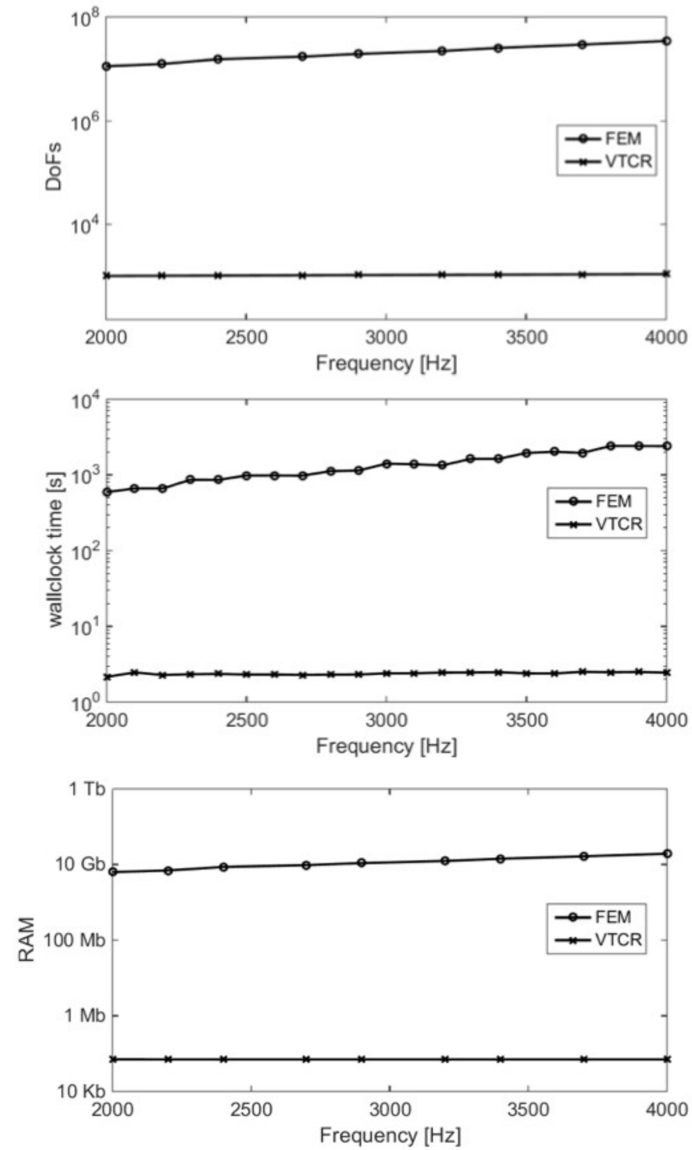
[Cattabiani 15]

Shells



frequency	2000	Hz
Young modulus	200	GPa
Poisson's ratio	0.3	
density	7800	kg/m^3
damping factor	0.01	

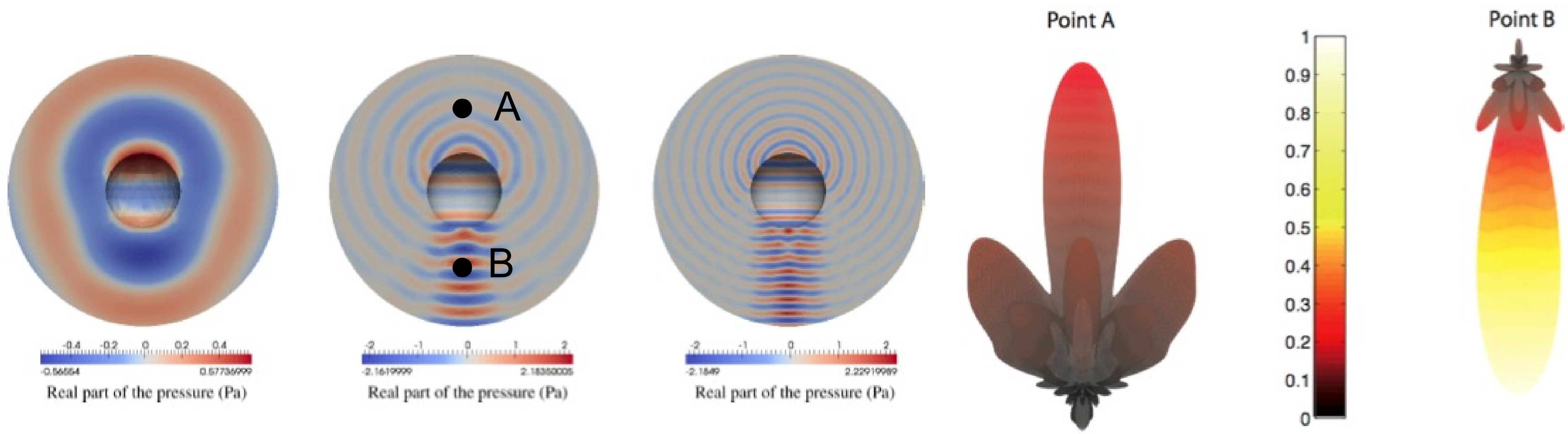


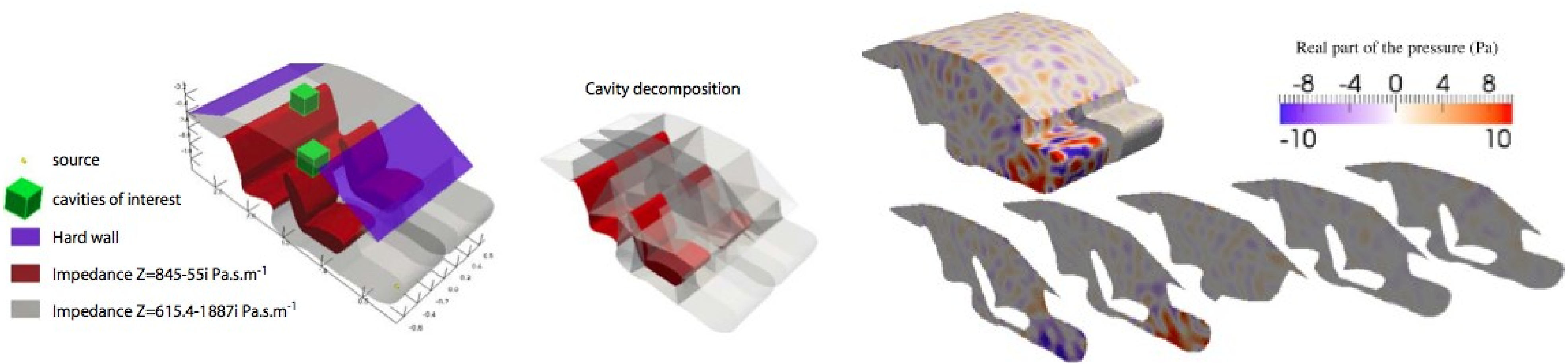


The Variational Theory of Complex Rays

[Kovalevsky et al 12]

3D acoustics

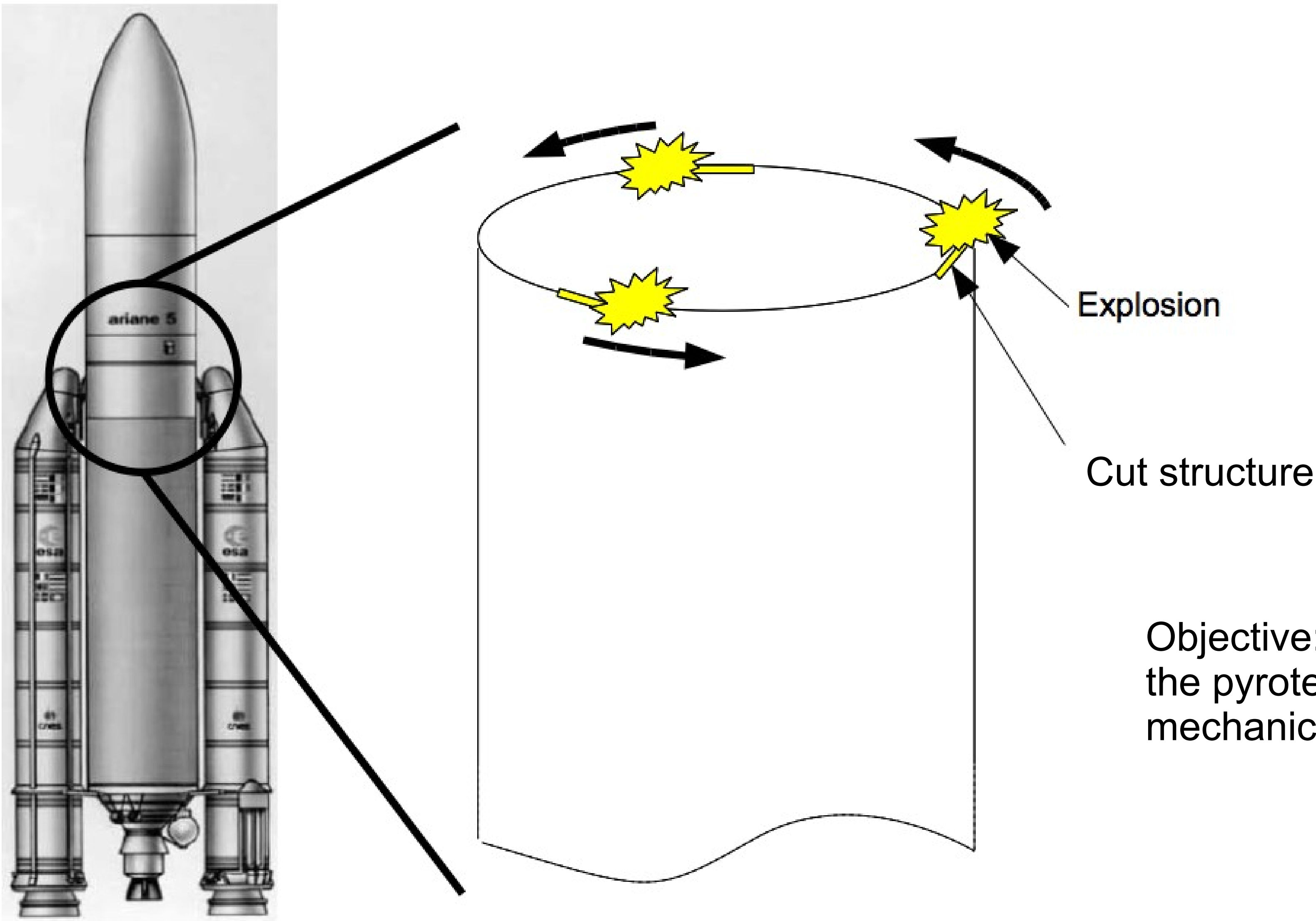




The Variational Theory of Complex Rays

[Riou 04]

Transient dynamics

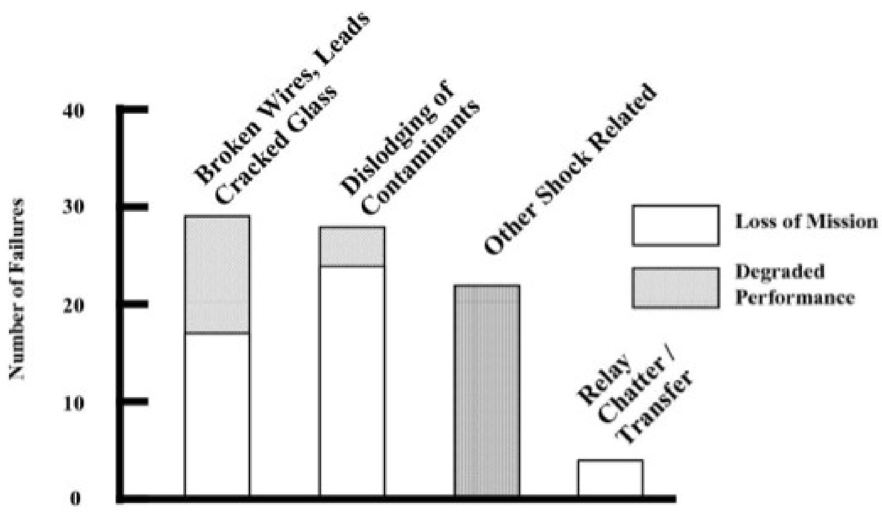


Objective: model the pyrotechnical mechanical action

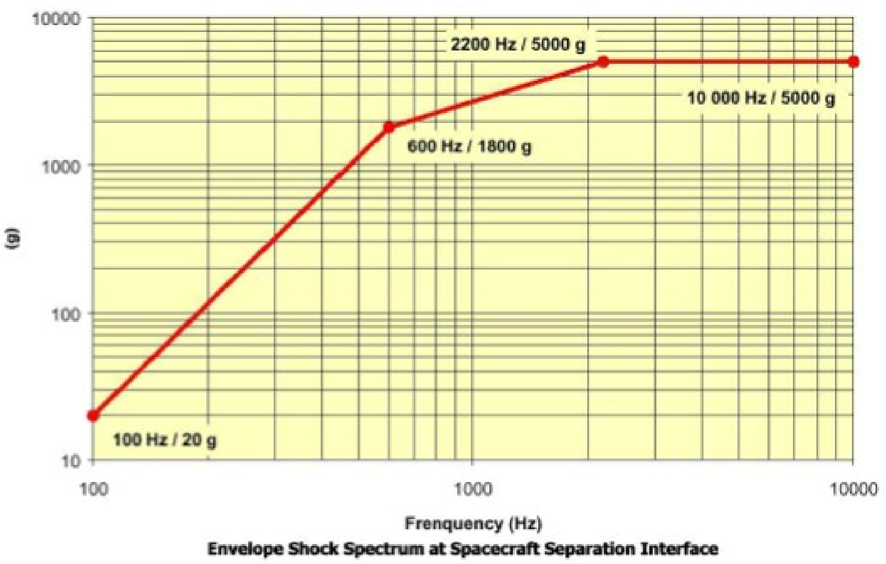
The Variational Theory of Complex Rays

[Riou 04]

Transient dynamics



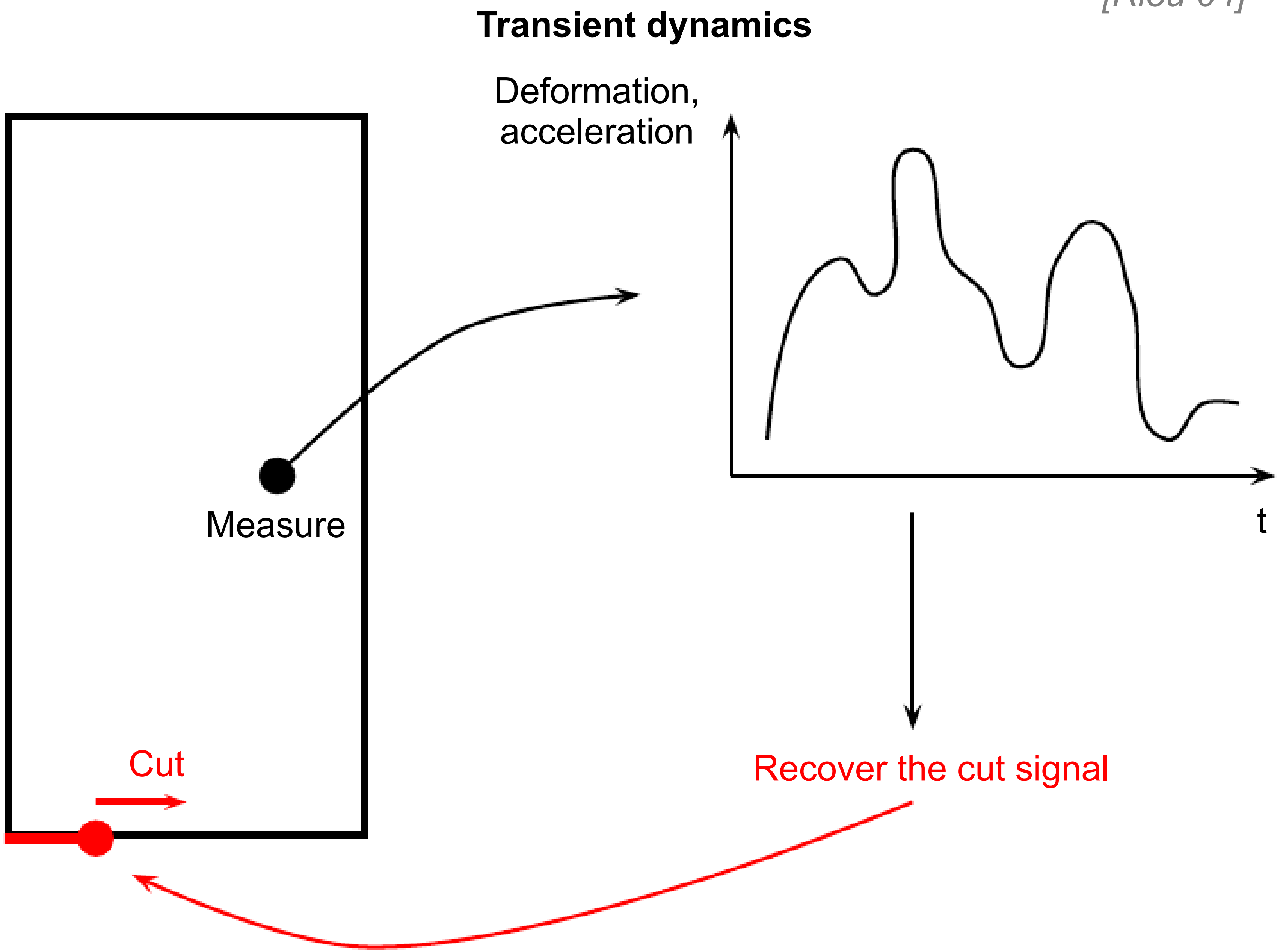
[Moening 85]



$$x_i(t)_{t\geq 0} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{x}_i(\omega)\mathrm{e}^{i\omega t}\mathrm{d}\omega$$

The Variational Theory of Complex Rays

[Riou 04]

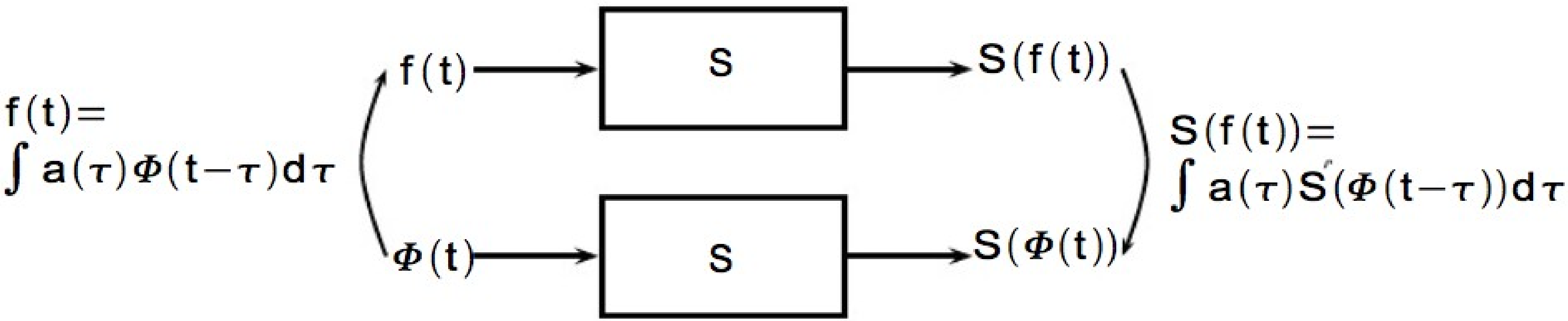


The Variational Theory of Complex Rays

[Riou 04]

Transient dynamics

Convolution / deconvolution

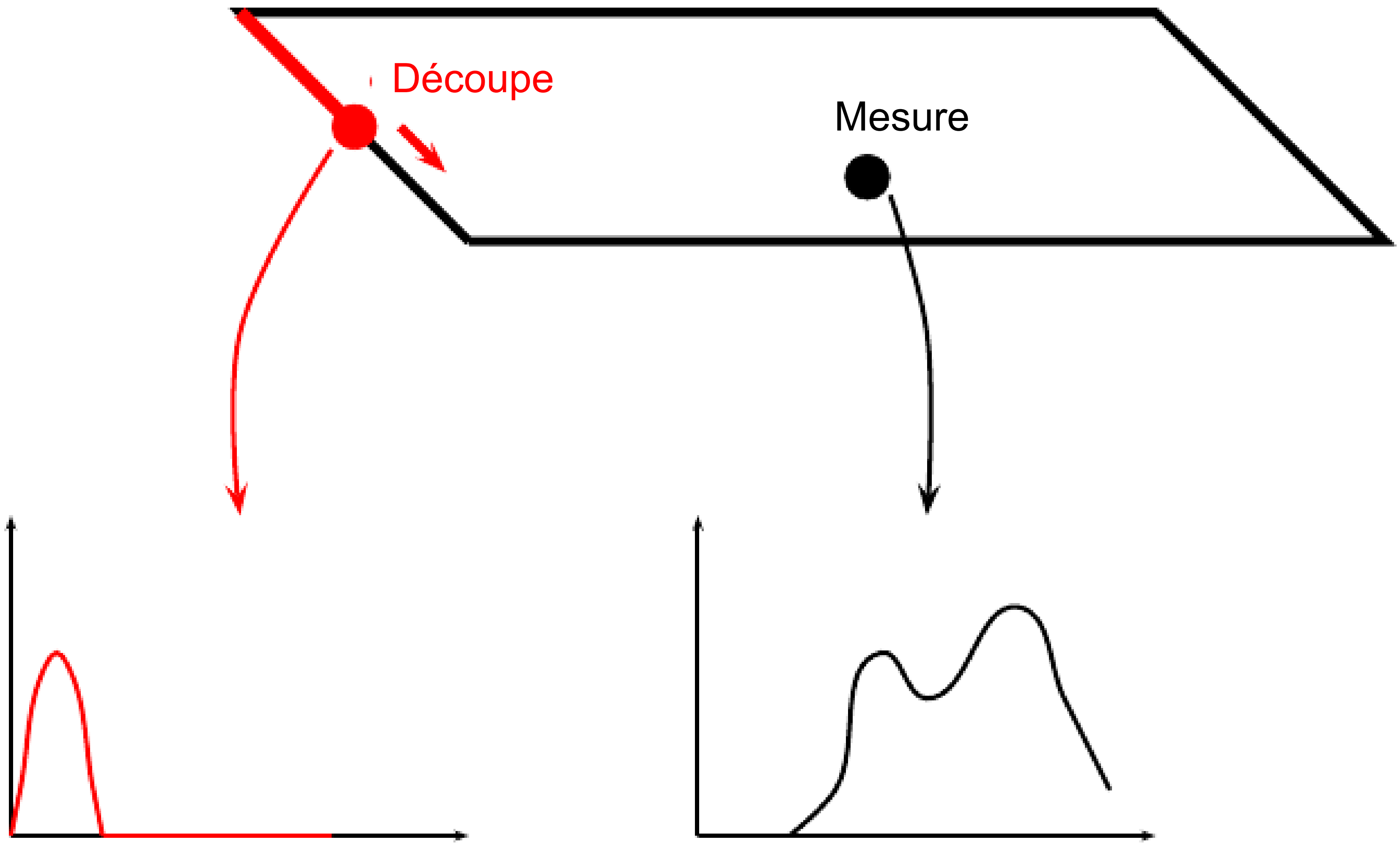


The Variational Theory of Complex Rays

[Riou 04]

Transient dynamics

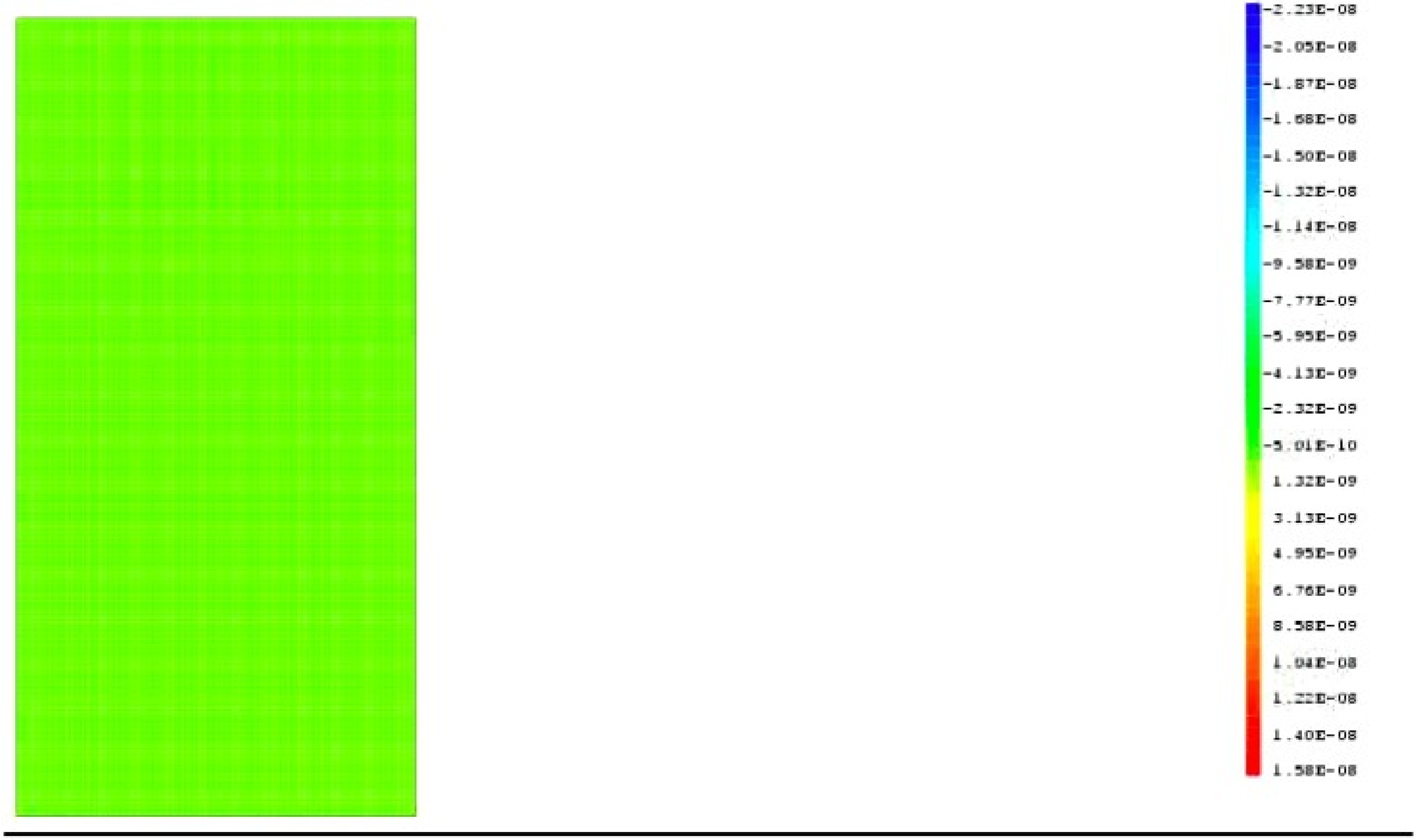
Tests



The Variational Theory of Complex Rays

[Riou 04]

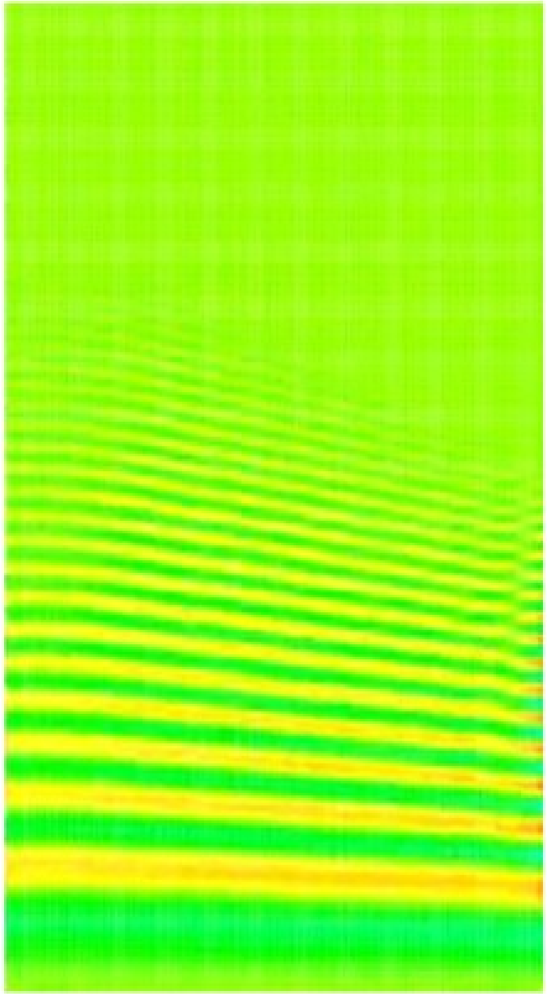
Transient dynamics

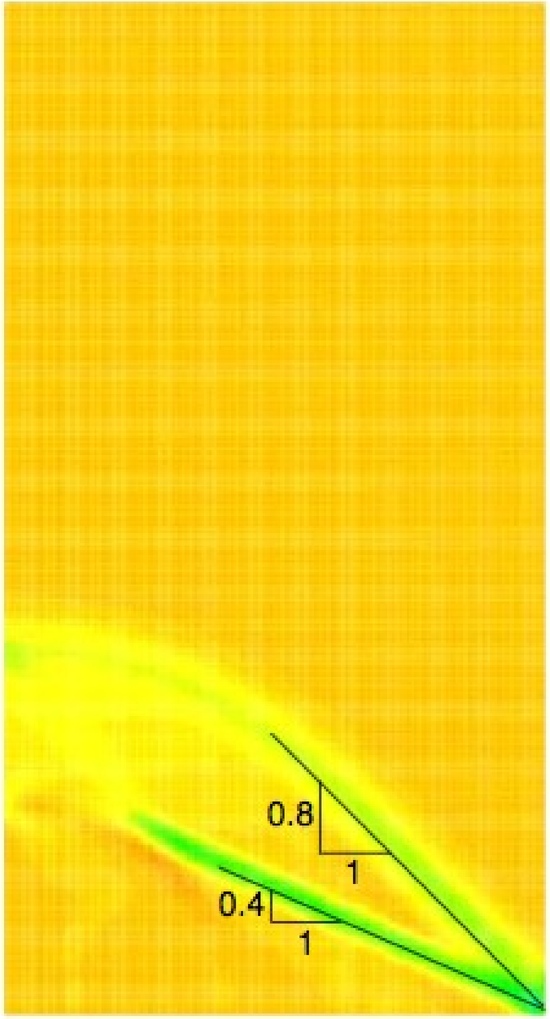


The Variational Theory of Complex Rays

[Riou 04]

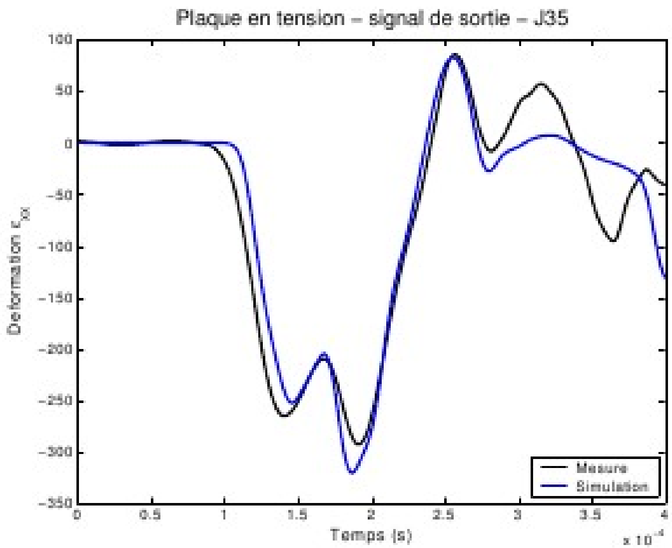
Transient dynamics

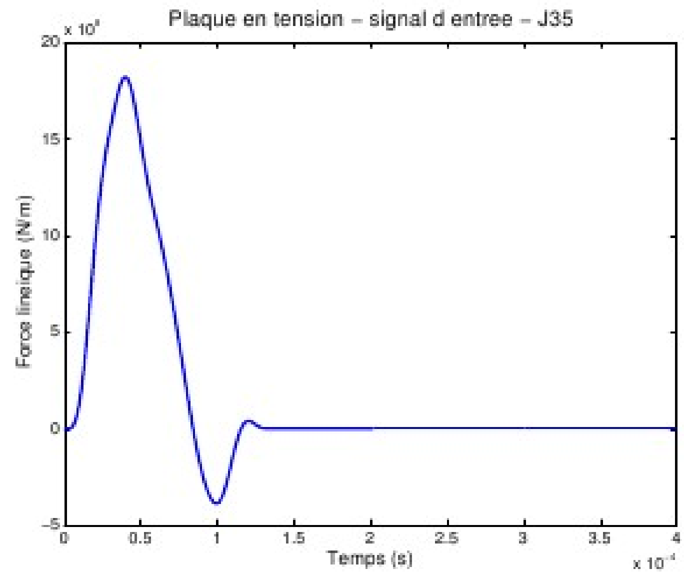


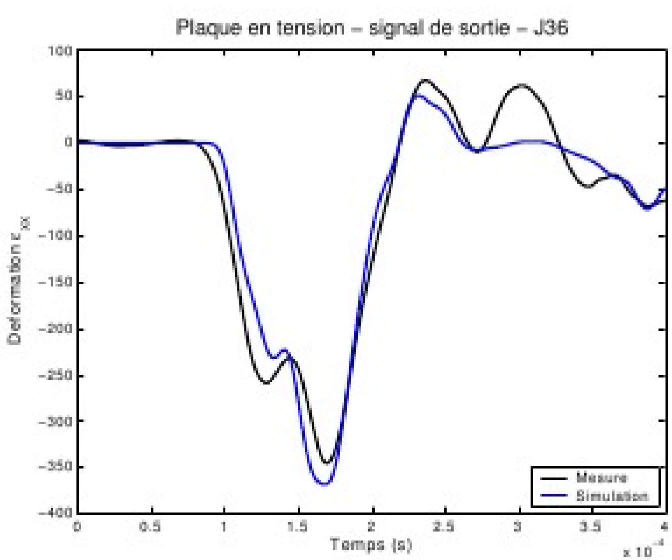


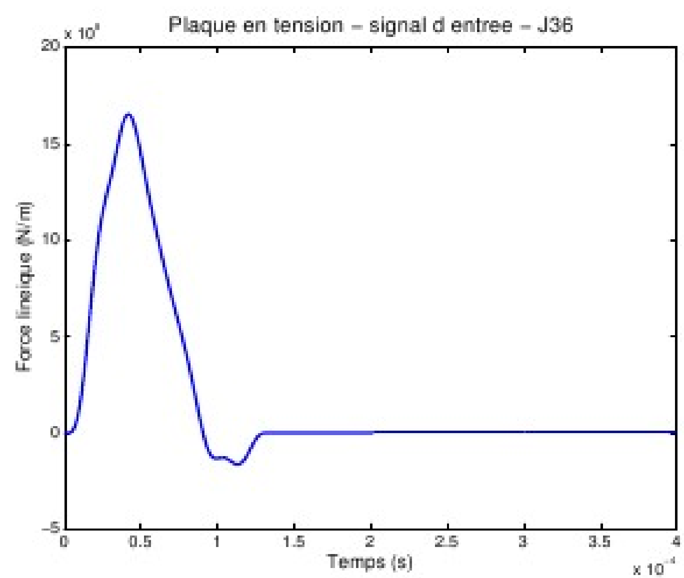
[Riou 04]

Transient dynamics





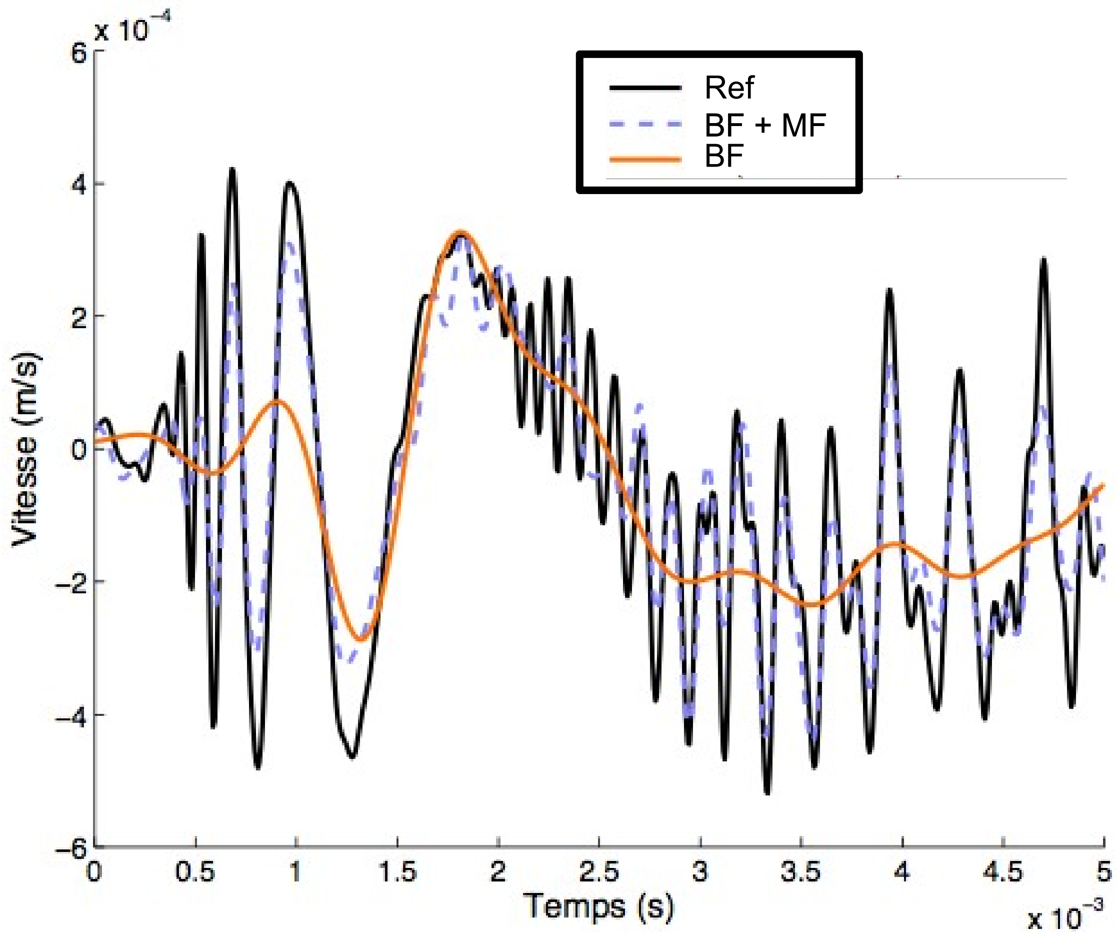




The Variational Theory of Complex Rays

[Chevreuil 08]

Transient dynamics



The Variational Theory of Complex Rays

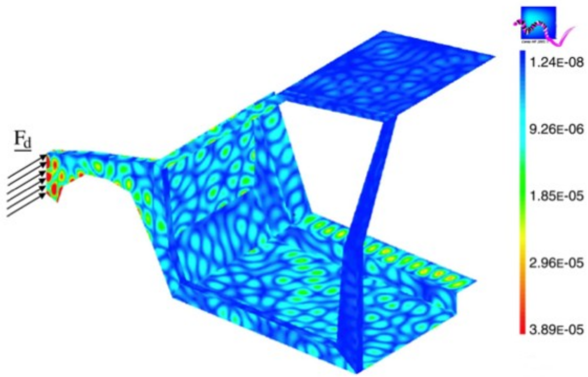
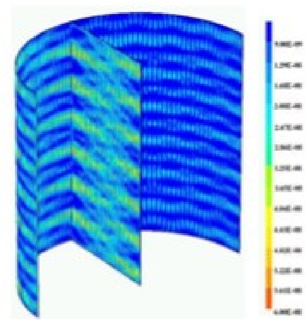
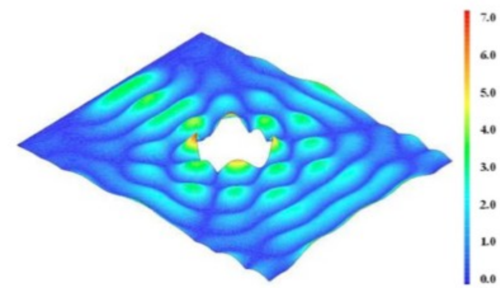
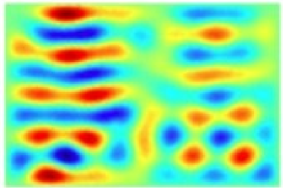


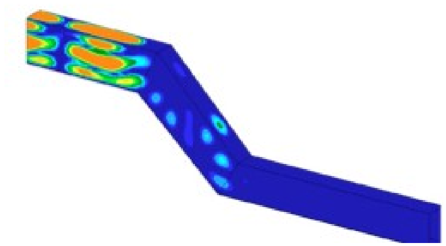
Plate structure

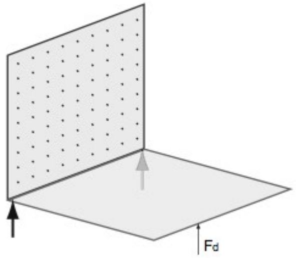

Shell assemblies



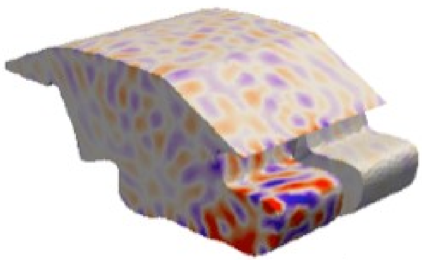
Heterogeneities

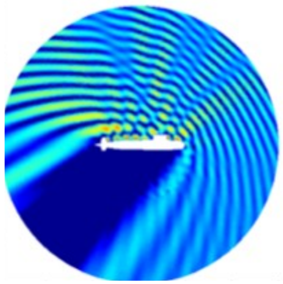

Composite material


Transient dynamics



Complex joints


3D acoustics


Scattering

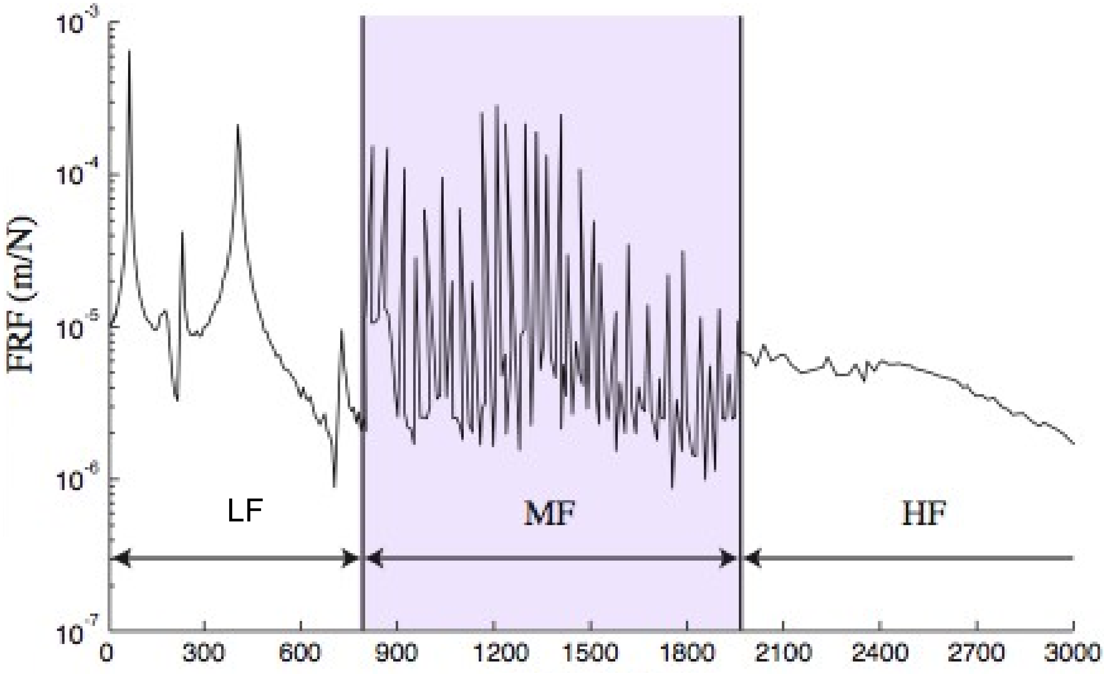
See [Rouch et al., 2002], [Riou et al., 2004], [Blanc et al., 2007], [Chevreuil et al., 2007], [Dorival et al., 2007], [Kovalevsky et al. 2012]

Trefftz methods

Can be widely used in midfrequency.
Preserve the rapid scale.
Few dofs.
Relation with the physics.
Very good efficiency.
No a priori limitation.
Need care to computational difficulties.

All the same conclusions can be drawn from DEM, LSM, PUM, UWVF, VTCR, WBM, ...

Mid frequencies



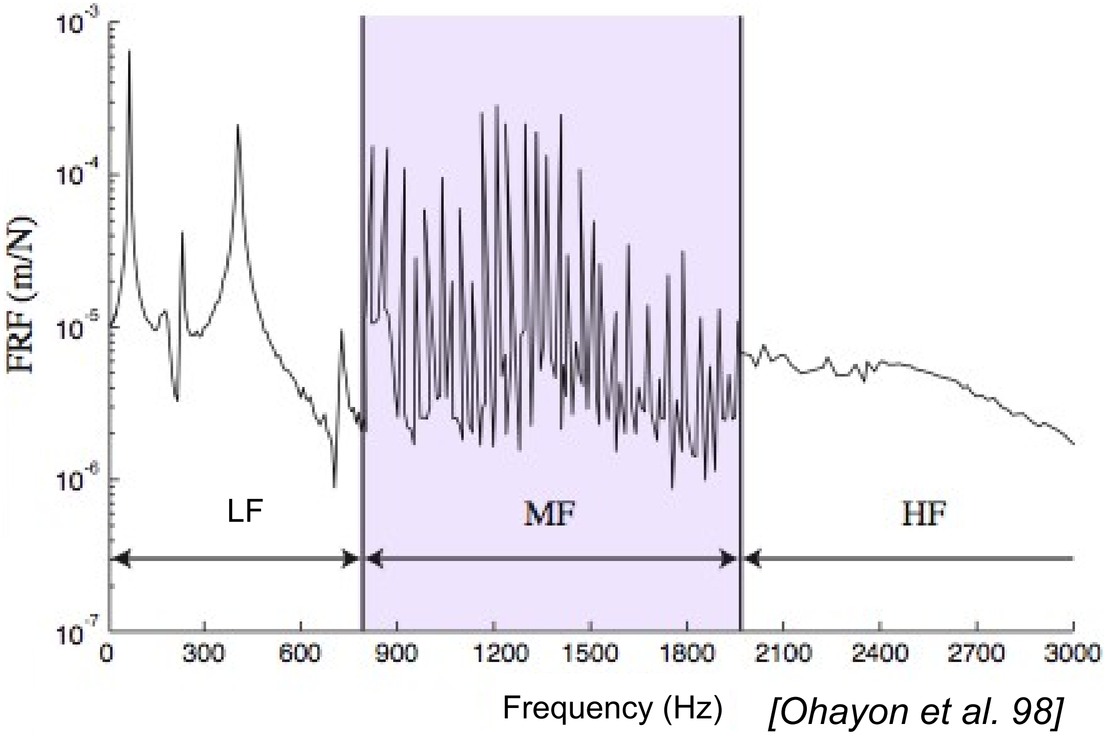
Frequency (Hz) *[Ohayon et al. 98]*

Car structure 300-3000 Hz
Acoustics noise *[Fahy 03]*

Sub marine 40-400 Hz
Detection problem *[Crocker 98]*

Satellite 200-2000 Hz
Vibrational ambiance on equipments
[Krammer et al. 08]

Mid frequencies



[Ohayon et al. 98]

Car structure 300-3000 Hz
Acoustics noise *[Fahy 03]*

Sub marine 40-400 Hz
Detection problem *[Crocker 98]*

200-2000 Hz
iance on equipments
[Krammer et al. 08]

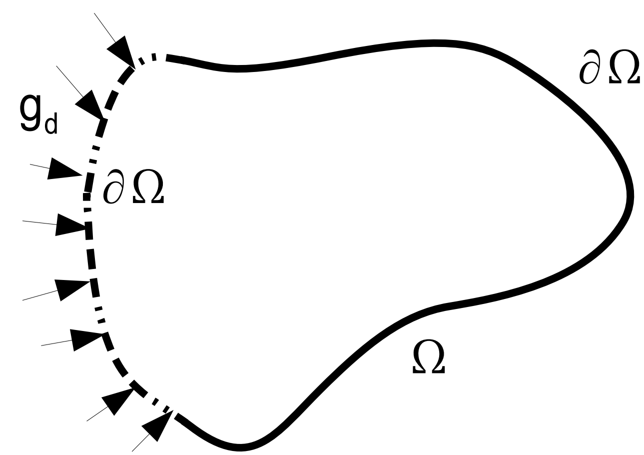
System complexity

System complexity => need for hybrid methods mixing different representations

« Weak Trefftz » method

- Weak Trefftz method (acoustics example)

$$\begin{cases} \text{Find } u=\{u_E\}_{E\in\mathbf{E}} \text{ such that} \\ \Delta u + k^2 u = 0 \in \Omega_E \\ \mathbf{q_u}.\mathbf{n} + Zu = g_d \text{ on } \partial\Omega \\ \{\mathbf{q_u}.\mathbf{n}\}_{E,E'} = 0 \text{ and } [u]_{E,E'} = 0 \text{ on } \Gamma_{E,E'} \end{cases}$$



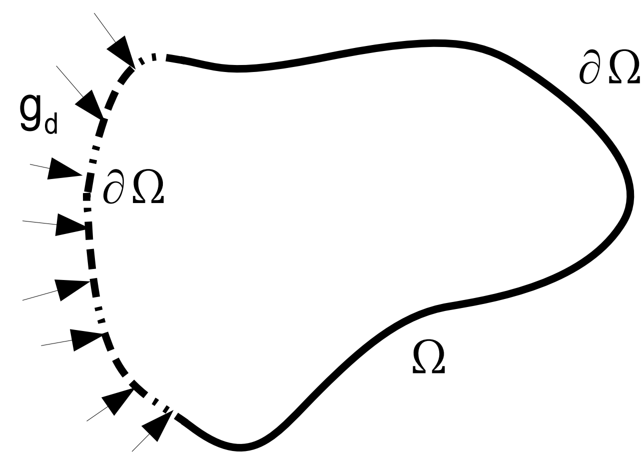
- Variational formulation

$$\begin{cases} \text{Find } u\in U \text{ such that} \\ a(u,v) = l(v) \quad \forall v\in U_0 \end{cases}$$

« Weak Trefftz » method

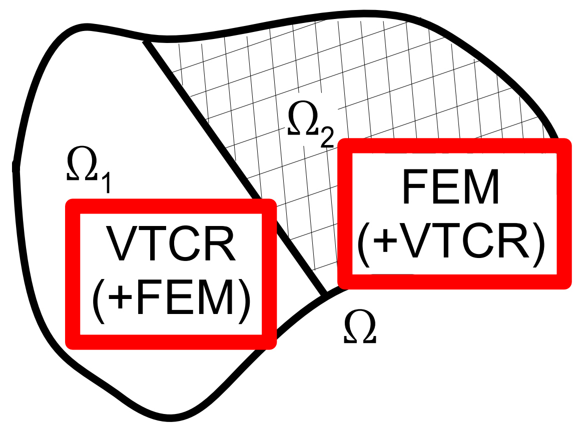
- Weak Trefftz method (acoustics example)

$$\begin{cases} \text{Find } u=\{u_E\}_{E\in\mathbf{E}} \text{ such that} \\ \Delta u+k^2u=0 \in \Omega_E \\ \mathbf{q_u}.\mathbf{n}+Zu=g_d \text{ on } \partial\Omega \\ \{\mathbf{q_u}.\mathbf{n}\}_{E,E'}=0 \text{ and } [u]_{E,E'}=0 \text{ on } \Gamma_{E,E'} \end{cases}$$



- Variational formulation

$$\begin{cases} \text{Find } u\in U \text{ such that} \\ a(u,v)=l(v) \;\; \forall v\in U_0 \end{cases}$$



Two different approximations

FEM formulation inadequate for VTCR formulation

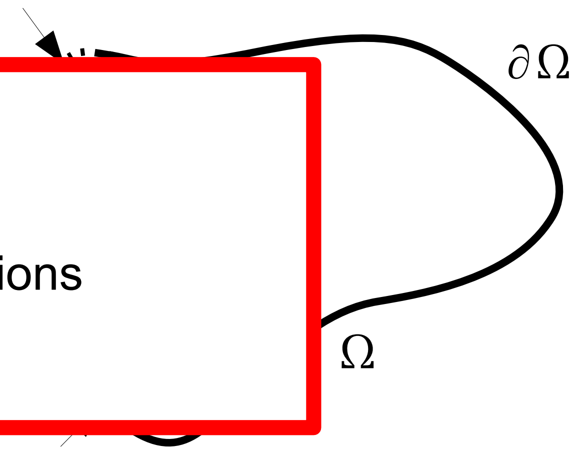
Trefftz-TVRC inadequate for FEM formulation

« Weak Trefftz » method

- Weak Trefftz method (acoustics example)

$$\begin{cases} \text{Find } u=\{u_E\} \\ \Delta u + k^2 u = 0 \in \\ \mathbf{q_u}.\mathbf{n} + Zu = g_d \\ \{\mathbf{q_u}.\mathbf{n}\}_{E,E'} = 0 \end{cases}$$

Need for new
"Weak Trefftz" formulations



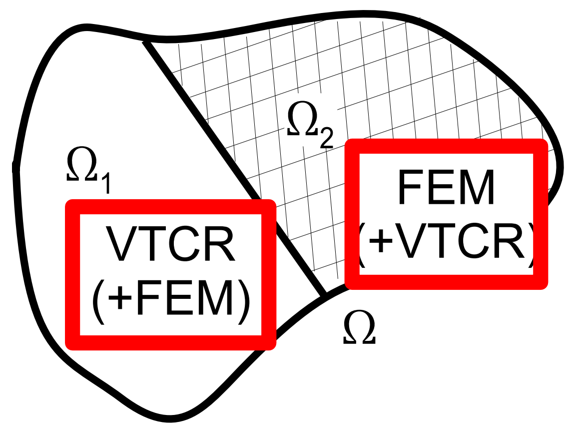
- Variational formulation

$$\begin{cases} \text{Find } u \in U \text{ such that} \\ a(u,v) = l(v) \quad \forall v \in U_0 \end{cases}$$

Two different approximations

FEM formulation inadequate for VTCR formulation

Trefftz-TVRC inadequate for FEM formulation



« Weak Trefftz » method

- Weak Trefftz method (acoustics example)

Find $u=\{u_E\}_{E\in\mathbf{E}}$ such that

$$\begin{cases} \Delta u+k^2u=0 \in \Omega_E \\ \mathbf{q_u}.\mathbf{n}+Zu=g_d \text{ on } \partial\Omega \\ \{\mathbf{q_u}.\mathbf{n}\}_{E,E'}=0 \text{ and } [u]_{E,E'}=0 \text{ on } \Gamma_{E,E'} \end{cases}$$

- Variational formulation

Find $u\in U$ such that

$$a(u,v)=l(v) \quad \forall v\in U_0$$

$$\sum_{E,E'}\int_{\Gamma_{E,E'}}\left(\frac{1}{2}\{\mathbf{q_u}.\mathbf{n}\}_{E,E'}\{\tilde{v}\}_{E,E'} - \frac{1}{2}[\tilde{\mathbf{q}}_{\mathbf{v}}.\mathbf{n}]_{E,E'}[u]_{E,E'}\right)dS$$
$$+\sum_E\int_{\partial\Omega}(\mathbf{q_u}.\mathbf{n}+Zu-g_d)\tilde{v}\,dS$$
$$-\sum_E\int_{\Omega_E}(\Delta u+k^2u)\tilde{v}\,d\Omega=0 \quad \forall v\in U_0$$

a(.,.) et l(.) are the bilinear and linear forms equivalent to all the equations of the initial problem

« Weak Trefftz » method

- Weak Trefftz method (acoustics example)

Find $u=\{u_E\}_{E\in \mathbf{E}}$ such that

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- Variational formulation

Find $u\in U$ such that

$$a(u,v) = l(v) \quad \forall v \in U_0$$

$$\sum_{E,E'} \int_{\Gamma_{E,E'}} \left(\frac{1}{2} \{\mathbf{q_u}.\mathbf{n}\}_{E,E'} \{\tilde{v}\}_{E,E'} - \frac{1}{2} [\tilde{\mathbf{q}}_\mathbf{v}.\mathbf{n}]_{E,E'} [u]_{E,E'} \right) dS + \sum_E \int_{\partial\Omega} (\mathbf{q_u}.\mathbf{n} + Zu - g_d)\tilde{v}\, dS - \sum_E \int_{\Omega_E} (\Delta u + k^2 u)\tilde{v}\, d\Omega = 0 \quad \forall v \in U_0$$

New term in addition to the VTCR classic formulation

a(.,.) et l(.) are the bilinear and linear forms equivalent to all the equations of the initial problem

« Weak Trefftz » method

- Weak Trefftz method (acoustics example)

$$\begin{cases} \text{Find } u=\{u_E\}_{E\in \mathbf{E}} \text{ such that} \\ \Delta u + k^2 u = 0 \in \Omega_E \\ \mathbf{q_u}.\mathbf{n} + Zu = g_d \text{ on } \partial\Omega \\ \{\mathbf{q_u}.\mathbf{n}\}_{E,E'} = 0 \text{ and } [u]_{E,E'} = 0 \text{ on } \Gamma_{E,E'} \end{cases}$$

- Variational formulation

$$\begin{cases} \text{Find } u\in U \text{ such that} \\ a(u,v) = l(v) \quad \forall v\in U_0 \end{cases}$$

$$\sum_{E,E'} \int_{\Gamma_{E,E'}} \left(\frac{1}{2}\{\mathbf{q_u}.\mathbf{n}\}_{E,E'}\{\tilde{v}\}_{E,E'} - \frac{1}{2}[\tilde{\mathbf{q}}_\mathbf{v}.\mathbf{n}]_{E,E'}[u]_{E,E'} \right) dS + \sum_E \int_{\partial\Omega} (\mathbf{q_u}.\mathbf{n} + Zu - g_d)\tilde{v}\, dS - \sum_E \int_{\Omega_E} (\Delta u + k^2 u)\tilde{v}\, d\Omega = 0 \quad \forall v \in U_0$$

Governing equation weaken (origin of the « Weak Trefftz » name)

a(.,.) et l(.) are the bilinear and linear forms equivalent to all the equations of the initial problem

« Weak Trefftz » method

- Weak Trefftz method (acoustics example)

Find $u=\{u_E\}_{E\in\mathbf{E}}$ such that

$$\Delta u+k^2u=0 \in \Omega_E$$
$$\mathbf{q_u}.\mathbf{n}+Zu=g_d \text{ on } \partial\Omega$$
$$\{\mathbf{q_u}.\mathbf{n}\}_{E,E'}=0 \text{ and } [u]_{E,E'}=0 \text{ on } \Gamma_{E,E'}$$

- Variational formulation

Find $u\in U$ such that

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$$+\sum_E\int_{\partial\Omega}(\mathbf{q_u}.\mathbf{n}+Zu-g_d)\tilde{v}\,dS$$
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Governing equation weaken (origin of the « Weak Trefftz » name)

a(.,.) et l(.) are t[...]he equations of the initial proble[...]

Any kind of approximation can be used (VTRC, FEM, other, ...)

« Weak Trefftz » method

- Weak Trefftz method (acoustics example)

$$\begin{cases} \text{Find } u=\{u_E\}_{E\in\mathbf{E}} \text{ such that} \\ \Delta u+k^2u=0 \in \Omega_E \\ \mathbf{q_u}.\mathbf{n}+Zu=g_d \text{ on } \partial\Omega \\ \{\mathbf{q_u}.\mathbf{n}\}_{E,E'}=0 \text{ and } [u]_{E,E'}=0 \text{ on } \Gamma_{E,E'} \end{cases}$$

- Variational formulation

$$\begin{cases} \text{Find } u\in U \text{ such that} \\ a(u,v)=l(v) \quad \forall v\in U_0 \end{cases}$$

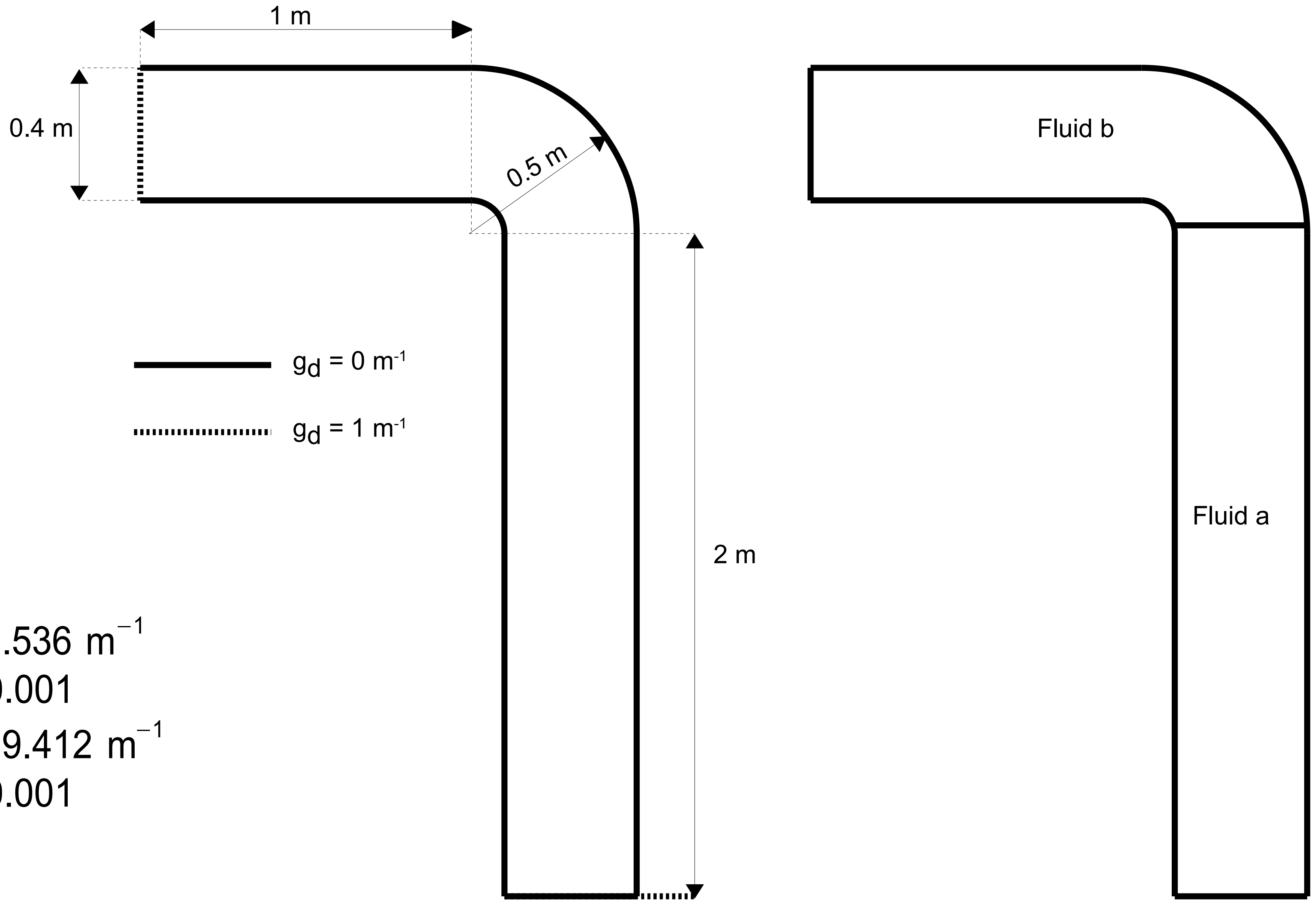
- Approximated solution

$$\begin{cases} U^h\subset U \\ \text{Fund } u^h\in U^h \text{ such that} \\ a(u^h,v^h)=l(v^h) \quad \forall v^h\in U_0^h \end{cases}$$

The weak Trefftz variational formulation is equivalent to the initial problem.

The discretized problem has a unique solution

« Weak Trefftz » method



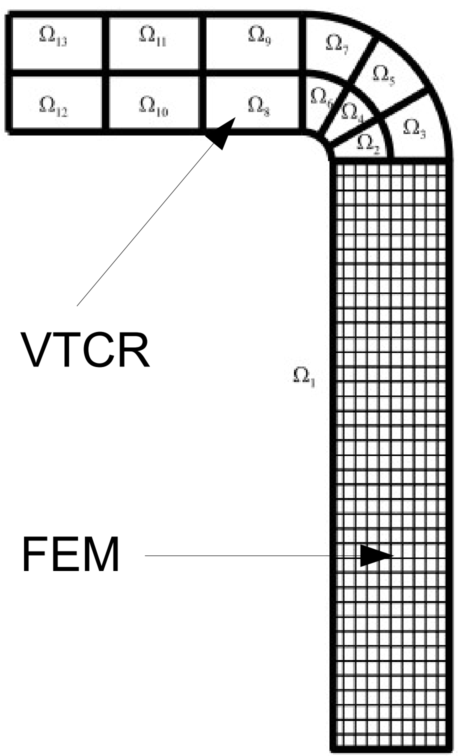
$k_a = 6.536\ m^{-1}$

$\eta_a = 0.001$

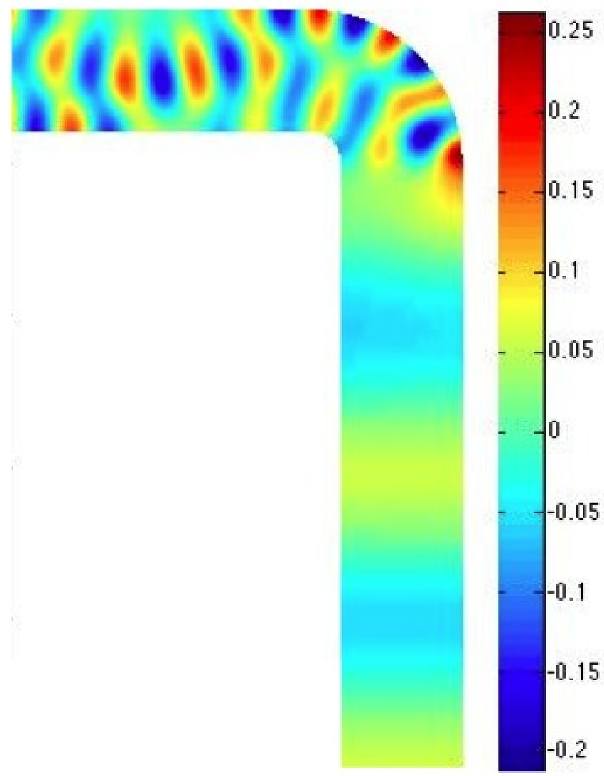
$k_b = 29.412\ m^{-1}$

$\eta_b = 0.001$

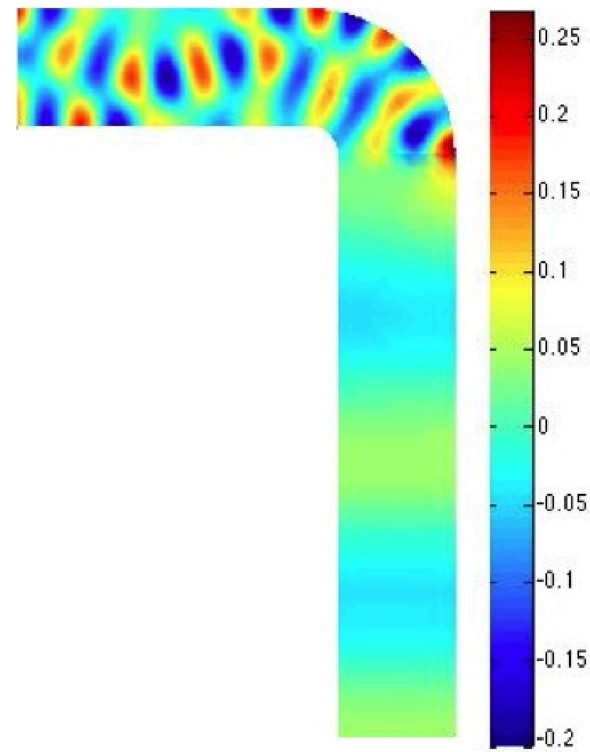
Discretization



Reference solution (FEM)



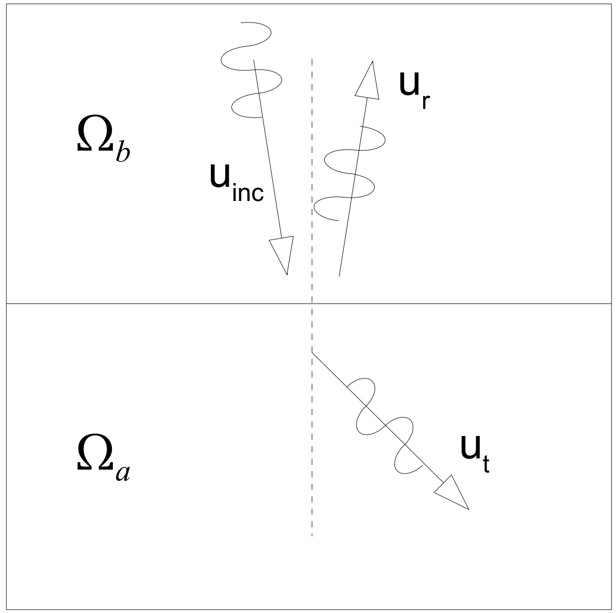
Weak Trefftz solution

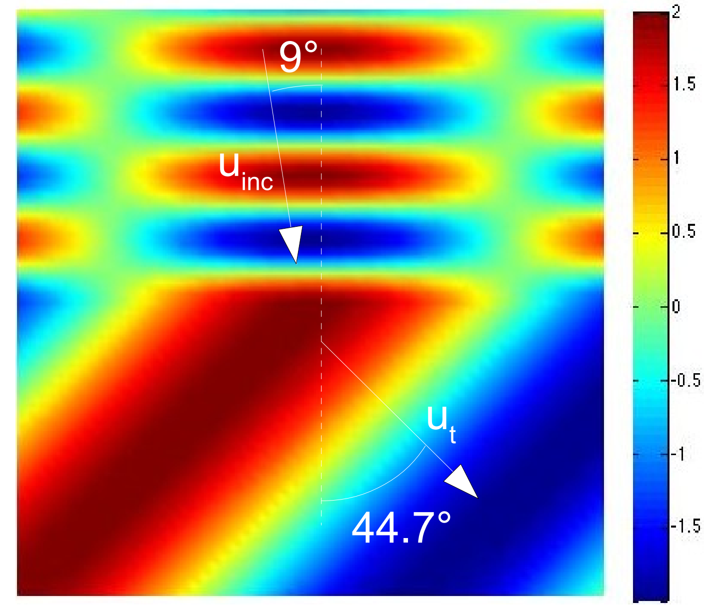


« Weak Trefftz » method

1 m x 1 m square

$k_a = 6.536\ m^{-1}$
$\eta_a = 0.001$
$k_b = 29.412\ m^{-1}$
$\eta_b = 0.001$

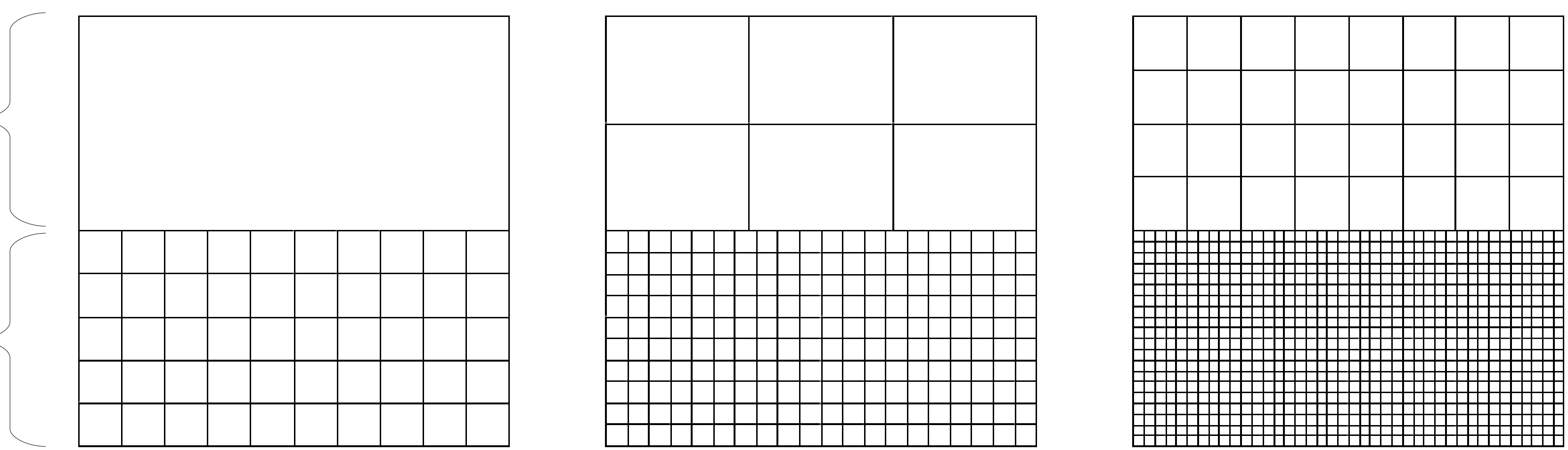




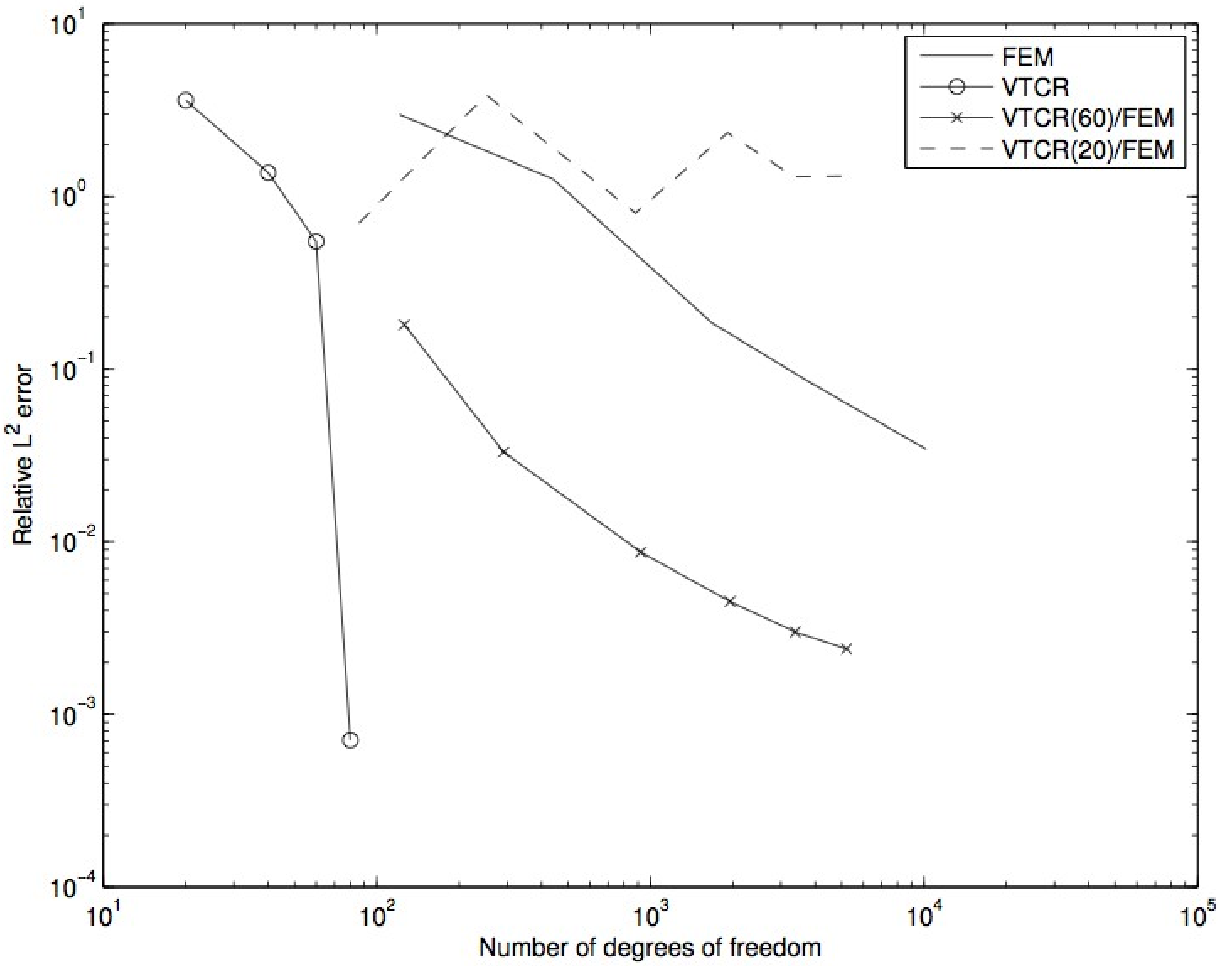
Discretizations

VTCR

FEM



« Weak Trefftz » method



Conclusions: convergence rate of the weak Trefftz method in agreement with the convergence rate of the discretizations

« Weak Trefftz » method

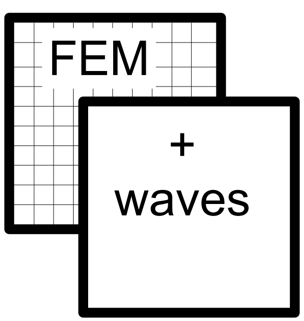
Non homogeneous problem with two scales

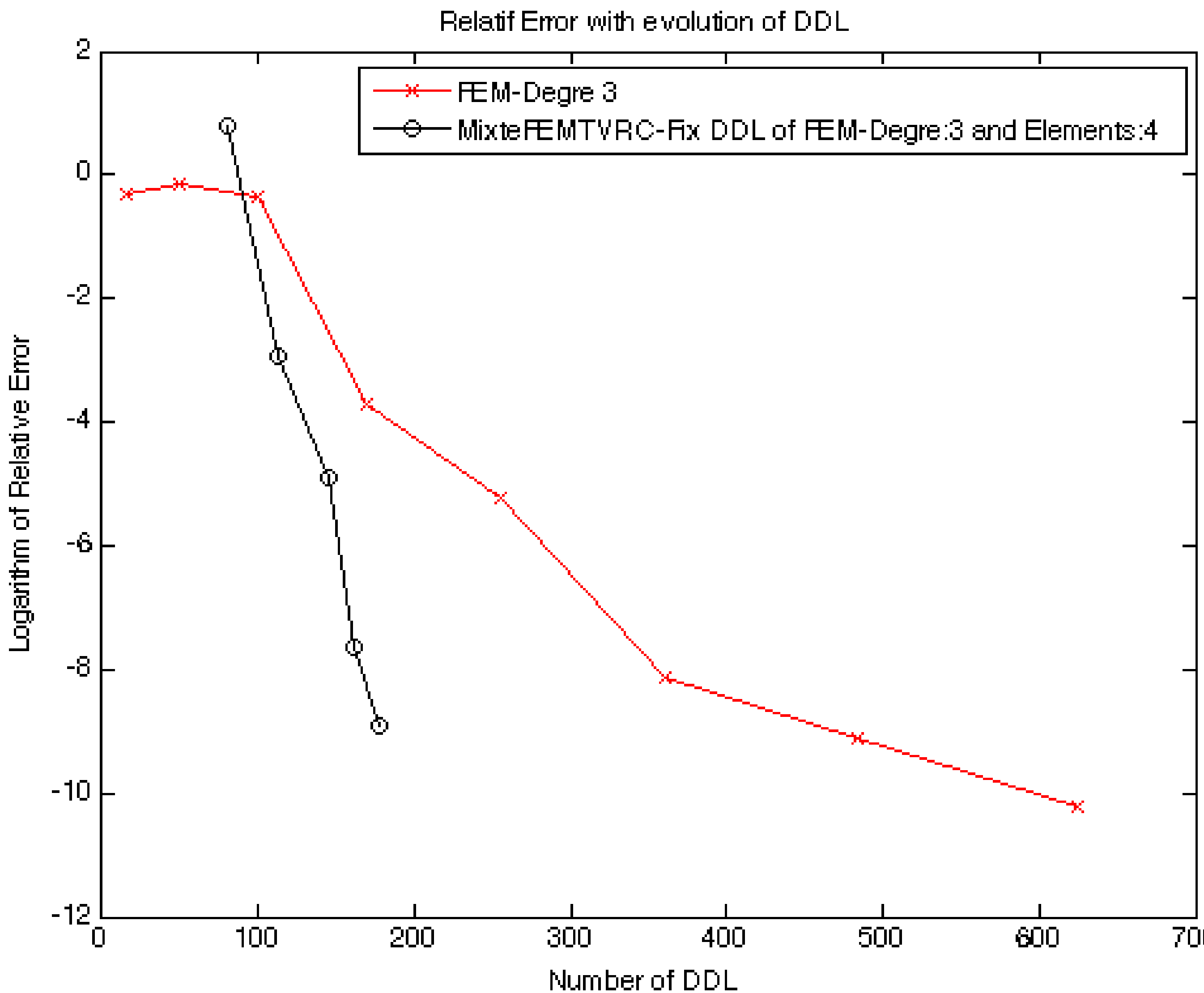
$$\Delta u + k^2 u = f_d \text{ in } \Omega$$

0,5 m x 0,5 m square domain

$$u_{ex} = \cos(k_1 \cos(\theta) x + k_1 \sin(\theta) y) + \cos(k_2 \cos(\theta) x + k_2 \sin(\theta) y)$$

$$k_1 = 35 \text{ m}^{-1}$$
$$k_2 = 10 \text{ m}^{-1}$$
$$\theta = 5 * pi / 180$$





« Weak Trefftz » method

Anisotropic homogeneous Helmholtz problem

$$\alpha u_{,xx}/k_x^2+\beta u_{,yy}/k_y^2+u=0 \text{ in } \Omega$$

0,5 m x 0,5 m square domain

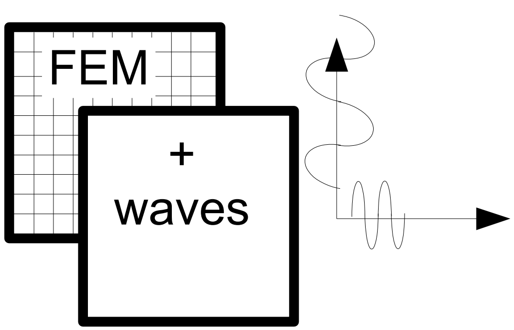
$$u_{ex}=e^{ik_x x/\sqrt{2\alpha}+ik_y y/\sqrt{2\beta}}$$

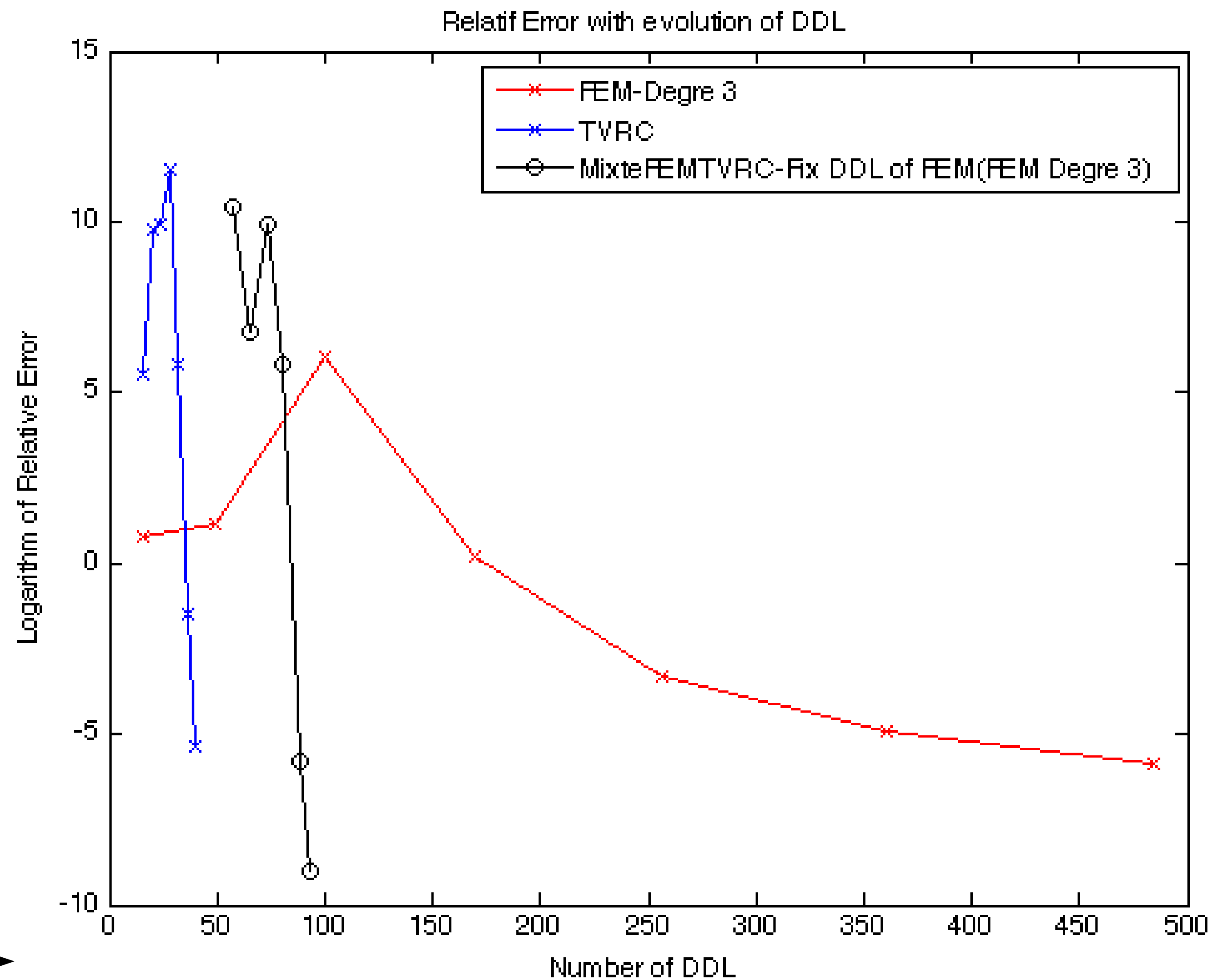
$$k_x=100 \text{ m}^{-1}$$

$$k_y=5 \text{ m}^{-1}$$

$$\alpha=2$$

$$\beta=0{,}5$$





Conclusion and prospects

- Conclusion

  Trefftz methods useful for mid frequency problems

  Trefftz methods extensively used, already

  Key points for industrial use: robustness, general use everywhere, mixing with other strategies

  Well suited to Trefftz way of thinking
  “A mathematical problem can only be said to be solved totally if - at the end - results can be produced in the form of numbers”

- Prospects

  Deeper analysis with physical aspects (relations medium-high frequency strategies)

  Development/extension of knowhow from different fields of mechanics

Thank you for your attention